

Zeros of Holant Problems: Locations and Algorithms

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We present fully polynomial-time (deterministic or randomised) approximation schemes for Holant problems, defined by a non-negative constraint function satisfying a generalised second-order recurrence modulo in a couple of exceptional cases. As a consequence, any non-negative Holant problem on cubic graphs has an efficient approximation algorithm unless the problem is equivalent to approximately counting perfect matchings, a central open problem in the area. This is in sharp contrast to the computational phase transition shown by two-state spin systems on cubic graphs. Our main technique is the recently established connection between zeros of graph polynomials and approximate counting.

CCS Concepts: • **Theory of computation** → **Design and analysis of algorithms; Randomness, geometry and discrete structures;**

Additional Key Words and Phrases: Approximate counting, zero-free region, Holant problem, holographic algorithms

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1 INTRODUCTION

Great progress has been made recently in the classification of counting problems. One major achievement is the full dichotomy for counting constraint satisfaction problems (CSPs) [5, 15],

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even with complex weights [7]. However, such a classification is for exact counting, and for approximation, even to move beyond some rather modest model seems quite difficult.

Holant problems [9] are a framework of expressing counting problems motivated by Valiant's holographic algorithms [42]. The "Holant" is a partition function on graphs where edges are variables and vertices are constraint functions. The benefit of this choice is the ability to express problems like perfect matchings, which are provably not expressible in certain CSP-like vertex models [12, 16, 37]. We parameterise Holant problems by the set of constraint functions that can be put on vertices. Similar to the success of classifying counting CSPs, exact classifications have been obtained for Holant problems defined by any set of complex-weighted symmetric Boolean functions [8] and progresses have been made towards classifying more general Holant problems [2, 10, 27].

In this article, we make progress towards understanding the complexity of approximating symmetric Boolean Holant problems with non-negative weights. Let $G = (V, E)$ be a graph, $\pi : V \rightarrow \mathcal{F}$ be an assignment from the set of vertices V to a set of functions \mathcal{F} , and $f_v = \pi(v)$ is the constraint function $\{0, 1\}^{\deg(v)} \rightarrow \mathbb{C}$ associated with the vertex v . The "Holant" is defined as follows:

$$Z(G; \pi) := \sum_{\sigma \in \{0,1\}^E} \prod_{v \in V} f_v(\sigma|_{E(v)}), \quad (1)$$

where $E(v)$ is the set of adjacent edges of v , and $\sigma|_{E(v)}$ is the restriction of σ on $E(v)$. We use the shorthand $Z(G)$ or Z when G and π are clear from the context.

We call a Boolean constraint function f *symmetric*, if $f(\mathbf{x})$ depends only on the hamming weight $|\mathbf{x}|$ and is invariant under permutations of the indices. For a symmetric f of arity d , we associate it with a *signature* $[f_0, f_1, \dots, f_d]$, where $f_i = f(\mathbf{x})$ if $|\mathbf{x}| = i$. We may use the term "constraint function" and "signature" interchangeably. For example, if f is the "exact-one" function, namely, $f = [0, 1, 0, \dots, 0]$, then $Z(G)$ counts the number of perfect matchings in G ; and if f is the Boolean OR function, namely, $f = [0, 1, 1, \dots, 1]$, then $Z(G)$ counts the number of edge covers in G . The "reversal" of a symmetric f is the function $\bar{f} = [f_d, f_{d-1}, \dots, f_0]$.

We focus on a fairly expressive family of symmetric functions satisfying generalised second-order recurrences. More precisely, we say $f = [f_0, f_1, \dots, f_d]$ satisfies a generalised second-order recurrence, if there exist real constants $(a, b, c) \neq (0, 0, 0)$ such that $af_k + bf_{k+1} + cf_{k+2} = 0$ for all $0 \leq k \leq d-2$. Denote by $\text{Holant}(f)$ the computational problem of evaluating $Z(G)$ where every vertex is associated with the signature f . In particular, the input to $\text{Holant}(f)$ must be a d -regular graph, where d is the arity of f . Our main theorem is the following.

THEOREM 1. *Let $f = [f_0, f_1, \dots, f_d]$ be a symmetric constraint function of arity $d \geq 3$ satisfying generalised second-order recurrences, and $f_i \geq 0$ for all $0 \leq i \leq d$. There is a fully polynomial-time (deterministic or randomised) approximation algorithm for $\text{Holant}(f)$, unless, up to a non-zero factor, f or its reversal is in one of the following form:*

- (1) $[0, \lambda \sin \frac{\pi}{d}, \lambda^2 \sin \frac{2\pi}{d}, \dots, \lambda^i \sin \frac{i\pi}{d}, \dots, 0]$ for some $\lambda > 0$;
- (2) $[0, 1, 0, \lambda, 0, \dots, 0, \lambda^{\frac{d-2}{2}}, 0]$ if d is even, or $[0, 1, 0, \lambda, 0, \dots, 0, \lambda^{\frac{d-1}{2}}]$ if d is odd, for some $0 \leq \lambda < 1$.

Moreover, in case (2), approximating $\text{Holant}(f)$ is equivalent to approximately counting perfect matchings in general graphs.

We remark that the approximation complexity of case (1) remains open in general.

Understanding the complexity of signatures with second-order recurrences is the cornerstone in the exact counting classifications. Since satisfying first-order recurrences implies that the function is degenerate, these constraint functions are the first class satisfying a recurrence relation with non-trivial complexity. More concretely, this family includes many interesting special cases:

- Matchings and perfect matchings, whose constraint functions are $[1, 1, 0, 0, \dots, 0]$ and $[0, 1, 0, 0, \dots, 0]$, respectively, with $(a, b, c) = (0, 0, 1)$.
- Even subgraphs, whose constraint functions are $[1, 0, 1, 0, \dots]$ with $(a, b, c) = (1, 0, -1)$. More generally, we may put weights on even and odd degree vertices, and the constraint functions become $[x, y, x, y, \dots]$ for some $x, y \geq 0$.
- Edge covers, whose constraint functions are $[0, 1, 1, \dots, 1]$ with $(a, b, c) = (0, 1, -1)$.
- Fibonacci gates, namely, f of arity d such that $f_{i+2} = b f_{i+1} + f_i$ for all $0 \leq i \leq d - 2$.
- All ternary symmetric functions.

For approximate counting, polynomial-time approximation algorithms are known only for a few special cases, such as counting matchings [21], weighted even subgraphs [22], counting edge covers [26], and a weighted version of Fibonacci gates [29]. However, neither the Markov chain Monte Carlo approach [21, 22] (including its “winding” extension [20, 30]), nor the correlation decay approach [26, 29], appears to be powerful enough to handle all functions in this family. However, Theorem 1 covers almost all problems in this family, and some of the exceptional cases are shown to be equivalent to counting perfect matchings, a central open problem in approximate counting (see, for example, [14, 40] on partial progress and barriers). Efficient approximate counting algorithm for perfect matchings is only known in the bipartite case [23].

As a consequence, we have an algorithm for all non-negative Boolean Holants on cubic graphs, unless the problem is equivalent to counting perfect matchings.

THEOREM 2. *Let $f = [f_0, f_1, f_2, f_3]$ be a symmetric constraint function of arity 3 where $f_i \geq 0$ for all $0 \leq i \leq 3$. $\text{Holant}(f)$ has a fully polynomial-time (deterministic or randomised) approximation algorithm, unless f or its reversal, up to a non-zero factor, is $[0, 1, 0, \lambda]$ for some $0 \leq \lambda < 1$. In the exceptional case, approximating $\text{Holant}(f)$ is equivalent to approximately counting perfect matchings in general graphs.*

We remark that Theorem 2 is in sharp contrast to the computational phase transition phenomenon, as demonstrated by two-state spin systems on cubic graphs [17, 18, 25, 38, 39], even without external fields. For spin systems, a clear and sharp threshold between approximable and hard to approximate is established for the parameters of the system, whereas for Holant problems on cubic graphs, there seems to be no such transition irrespective of the value of the parameters (modulo the open case of approximately counting perfect matchings).

1.1 Our Techniques

Our algorithm combines a number of ingredients:

- Barvinok’s approach to approximate partition functions via Taylor expansions [3]. This approach was sharpened by Patel and Regts [31] to run within polynomial-time.
- To apply Barvinok’s approach, one has to have some rather precise knowledge of the zeros of the corresponding graph polynomials. For Holant problems, Ruelle [34–36] has developed a systematic approach of bounding the zeros of the partition function via analysing polynomials associated locally with vertices, under the disguise of “graph-counting polynomials.”
- On top of combining Ruelle’s and Barvinok’s approaches, we also employ holographic transformations a la Valiant [42], which is necessary to cover all cases in Theorem 1.

Although none of these ingredients is new, the main contribution of our work is to combine them together (with reworks if necessary), and thoroughly analyses the zeros of functions with generalised second-order recurrences. To be more specific, for a symmetric signature $f = [f_0, \dots, f_d]$

of arity d , define the “local” polynomial of f as

$$P_f(z) := \sum_{i=0}^d \binom{d}{i} f_i \cdot z^i. \quad (2)$$

We may also view $P_f(z)$ as the polynomial for a single vertex with d dangling edges. For some $\varepsilon > 0$, we call a polynomial $P(z)$ H_ε -stable, if $P(z) \neq 0$ as long as $\Re z > -\varepsilon$. Then one of our main technical tool (see Theorem 15) says that if $P_f(z)$ is H_ε -stable for some $\varepsilon > 0$, then a polynomial-time approximation algorithm exists for $\text{Holant}(f)$.

In general, to apply Barvinok’s method to approximate counting, one needs to deal with the zeros of the whole partition function, which is usually not an easy task. Previous applications appeal to some powerful tools such as the Lee-Yang theorem from statistical physics [28], or the resolution of a long-standing conjecture [32]. In contrast, our approach requires only analysing some low degree polynomials and is much easier to apply.

To go from Theorem 1 to Theorem 2, we also need to deal with cases not covered by Theorem 1, which cannot be solved using zeros of Holant problems. These exceptional cases are handled by the “winding” technique [20, 30] with Markov chains.

2 RUELLE’S METHOD ON ZEROS OF HOLANT PROBLEMS

Ruelle [34–36] (building upon the “Asano contraction” [1]) has developed a systematic approach to bound zeros of the so-called “graph-counting polynomials.” As we will see later, these polynomials coincide with unweighted Holant problems.

With a little abuse of notation, let $Z(G; f)$ be the partition function defined by (1) where $f_v = f$ for all $v \in V$, and stratify $Z(G; f)$ by the number of edges chosen as follows:

$$Z_k(G; f) := \sum_{\sigma \in \{0,1\}^E \text{ and } |\sigma|=k} \prod_{v \in V} f(\sigma|_{E(v)}). \quad (3)$$

Define $Z_k(G; \pi)$ similarly, and again, G and f may be omitted when they are clear from the context.

Let $|E| = m$. Then $Z = Z(G; f)$ can be rewritten as the evaluation of the polynomial

$$P_G(z) := \sum_{i=0}^m Z_i \cdot z^i \quad (4)$$

at $z = 1$. Namely, $Z = P_G(1)$. When f is a symmetric 0/1 function, then Equation (4) is the same as the “graph-counting” polynomial defined by Ruelle [36].

Ruelle’s method has two main ingredients. First, we want to relate zeros of a univariate polynomial with those of its polar form. For a polynomial $P(z) = \sum_{i=0}^{d'} a_i z^i$ of degree $d' \leq d$, its d th polar form with variables $\mathbf{z} = (z_1, \dots, z_d)$ is

$$\widehat{P}(\mathbf{z}) := \sum_{I \subseteq [d]} \frac{a_{|I|}}{\binom{d}{|I|}} z_I,$$

where $a_i = 0$ if $i > d'$, $[d]$ denotes $\{1, 2, \dots, d\}$, and for an index set I , $z_I = \prod_{i \in I} z_i$. For example, the polar form of $P_f(z)$ (recall Equation (2)) is

$$\widehat{P}_f(\mathbf{z}) := \sum_{I \subseteq [d]} f_{|I|} z_I.$$

The polar form $\widehat{P}(\mathbf{z})$ is the unique multi-linear symmetric polynomial of degree at most d' such that $\widehat{P}(z, z, \dots, z) = P(z)$. When $d' < d$, we view $P(z)$ as a degenerate case, and it has zeros at ∞ with multiplicity $d - d'$.

Let H be a set in \mathbb{C} and we use $\overline{H} = \{z \in \mathbb{C} \mid z \notin H\}$ to denote its complement. We say a polynomial $P(\mathbf{z})$ in $d \geq 1$ variables is H -stable if $P(\mathbf{z}) \neq 0$ whenever $z_1, \dots, z_d \in H$. We will be particularly interested in H_ε -stability where H_ε is the half-plane:

$$H_\varepsilon = \{z \in \mathbb{C} \mid \Re z > -\varepsilon\},$$

and $\varepsilon > 0$. The Grace-Szegő-Walsh coincidence theorem [19, 41, 43] has the following immediate consequence. See Reference [11] for the form we use here.

PROPOSITION 3. *A univariate polynomial $P(z)$ is H_ε -stable if and only if its polar form $\widehat{P}(z)$ is H_ε -stable.*

Proposition 3 actually applies to an arbitrary circular domain in \mathbb{C} , but we will only need it for H_ε .

The next ingredient is the Asano contraction [1], as extended by Ruelle [34].

PROPOSITION 4. *Let K_1 and K_2 be closed subsets of the complex plane \mathbb{C} , which do not contain 0. If the complex polynomial*

$$\alpha + \beta z_1 + \gamma z_2 + \delta z_1 z_2$$

can vanish only when $z_1 \in K_1$ or $z_2 \in K_2$, then

$$\alpha + \delta z$$

can vanish only when $z \in -K_1 \cdot K_2$ ($:= \{-a \cdot b \mid a \in K_1, b \in K_2\}$).

We refer interested readers to Reference [34] for a very elegant proof of Proposition 4.

Let the δ -strip of $[0, 1]$ be

$$\{z \in \mathbb{C} \mid |\Im z| \leq \delta \text{ and } -\delta \leq \Re z \leq 1 + \delta\}.$$

LEMMA 5. *For any $\varepsilon > 0$, the complement of $-\overline{H_\varepsilon} \cdot \overline{H_\varepsilon}$ contains a δ -strip of $[0, 1]$ for some $\delta > 0$ depending only on ε .*

PROOF. An equivalent way to write $\overline{H_\varepsilon}$ is

$$\overline{H_\varepsilon} = \left\{ \rho e^{i\theta} \mid \rho \geq -\frac{\varepsilon}{\cos \theta} \text{ for } \theta \in \left(\frac{\pi}{2}, \frac{3\pi}{2} \right) \right\}.$$

Thus,

$$\begin{aligned} -\overline{H_\varepsilon} \cdot \overline{H_\varepsilon} &= \left\{ \rho_1 \rho_2 e^{i(\theta_1 + \theta_2 + \pi)} \mid \rho_i \geq -\frac{\varepsilon}{\cos \theta_i} \text{ for } \theta_i \in \left(\frac{\pi}{2}, \frac{3\pi}{2} \right) \text{ and } i \in \{1, 2\} \right\} \\ &= \left\{ \rho e^{i(\theta_1 + \theta_2 + \pi)} \mid \rho \geq \frac{\varepsilon^2}{\cos \theta_1 \cos \theta_2} \text{ for } \theta_1, \theta_2 \in \left(\frac{\pi}{2}, \frac{3\pi}{2} \right) \right\} \\ &= \left\{ \rho e^{i\theta} \mid \rho \geq \frac{\varepsilon^2}{\left(\cos \frac{\theta - \pi}{2} \right)^2} \text{ for } \theta \in (0, 2\pi) \right\} \\ &= \left\{ \rho e^{i\theta} \mid \rho \geq \frac{2\varepsilon^2}{1 - \cos \theta} \text{ for } \theta \in (0, 2\pi) \right\}, \end{aligned}$$

where the third line is because $\cos \theta_1 \cos \theta_2$ is maximised at $\theta_1 = \theta_2$ if their sum is fixed. Thus, the complement of $-\overline{H_\varepsilon} \cdot \overline{H_\varepsilon}$ is

$$\overline{-\overline{H_\varepsilon} \cdot \overline{H_\varepsilon}} = \left\{ \rho e^{i\theta} \mid \rho < \frac{2\varepsilon^2}{1 - \cos \theta} \text{ for } \theta \in (0, 2\pi) \right\}.$$

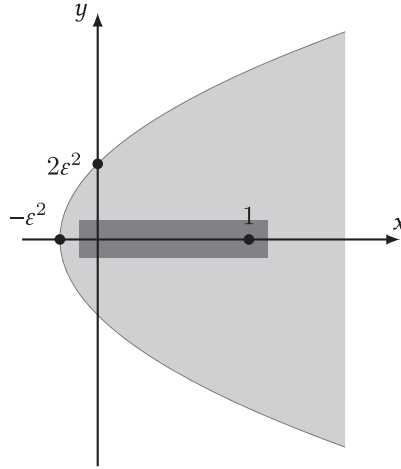


Fig. 1. $\overline{-H_\varepsilon} \cdot \overline{H_\varepsilon}$ (in light gray) and the δ -strip of $[0, 1]$ (in dark gray) for $\varepsilon = 0.5$.

It is easy to check that $\delta = \varepsilon^2/2$ suffices for the claim (See Figure 1 for an illustration of two regions). \square

Now, we are ready to state a very useful lemma.

LEMMA 6. *Let f be a symmetric signature of arity Δ and G be a Δ -regular graph. If the local polynomial $P_f(z)$ is H_ε -stable for some $\varepsilon > 0$, then the global polynomial $P_G(z)$ has no zero in the δ -strip of $[0, 1]$, where δ is a constant depending only on ε .*

PROOF. We construct $G = (V, E)$ as follows. Start with a collection of vertices $v \in V$, each with Δ dangling half-edges $(e_i^v)_{i \in [\Delta]}$. Call this graph G_0 , and connect dangling half-edges e_i^v and e_j^u sequentially for each edge $(u, v) \in E$. This gives a sequence of graphs $G_1, \dots, G_{|E|} = G$. The polynomial of G_0 is $P_{G_0}(z) = \prod_{v \in V} P_v(z)$, where $P_v = P_f$, and consider the multivariate version $\widehat{P}_{G_0}(z) = \prod_{v \in V} \widehat{P}_v(z^v)$, where $\widehat{P}_v = \widehat{P}_f$ and z^v denotes the local variables corresponding to v . Since $P_f(z)$ is H_ε -stable, by Proposition 3, $\widehat{P}_f(z)$ is as well, and so is $\widehat{P}_{G_0}(z)$. Suppose from G_i to G_{i+1} , e_i^v is connected with e_j^u . Then the transformation from \widehat{P}_{G_i} to $\widehat{P}_{G_{i+1}}$ is exactly the Asano contraction as in Proposition 4 applied to z_i^v and z_j^u . At the end of this procedure, we obtain G and the polynomial $\widehat{P}_G(z)$ does not vanish on the complement of $\overline{-H_\varepsilon} \cdot \overline{H_\varepsilon}$. It implies that the same is true for the univariate $P_G(z)$. By Lemma 5, the complement of $\overline{-H_\varepsilon} \cdot \overline{H_\varepsilon}$ contains a δ -strip of $[0, 1]$, and this δ depends only on ε . \square

We note that it is necessary to have some slack ε in Lemma 6. One example is counting even subgraphs, namely, the constraint f is $[1, 0, 1, 0, \dots]$. Although all zeros of P_f lie on the imaginary axis, the zeros of $P_G(z)$ can in fact be dense on the unit circle. To see this, let G be a cycle of length n . Then $P_G(z) = 1 + z^n$ as there are only two even subgraphs. The zeros thereof are dense on the unit circle as n varies.

A related result obtained by Regts [33] asserts that if $P_f(z)$ is sufficiently close to $(x+1)^d$, then the global polynomial $P_G(z)$ has no zero in certain disks around the origin. We refer the readers to Reference [33] for detailed statements. Such a zero-free region also implies the existence of approximation algorithms, and cannot be directly compared with Lemma 6.

Lemma 6 can be generalised to a set of functions by following the proof of Lemma 6, if there is an $\varepsilon > 0$ such that all of the local polynomials are H_ε -stable. A univariate polynomial is called

Hurwitz stable if all of its zeros are in the open left half-plane. For a fixed f , clearly if $P_f(z)$ is Hurwitz stable, then there is some $\varepsilon > 0$ such that $P_f(z)$ is H_ε -stable. However, Hurwitz stability is not enough to derive the same conclusion of Lemma 6 for an infinite set of functions.

3 BARVINOK'S ALGORITHM

Our interest in Ruelle's method is due to the algorithmic approach developed by Barvinok [3, Section 2]. It roughly states that if a polynomial $P(z) = \sum_{i=0}^n c_i z^i$ of degree n is zero-free in a strip containing $[0, 1]$, then $P(1)$ can be $(1 \pm \varepsilon)$ -approximated using c_0, \dots, c_k for some $k = O(\log \frac{n}{\varepsilon})$.

The basic idea is to truncate the Taylor expansion of $\log P(z)$ at $z = 0$. Let $g(z) := \log P(z)$ and for $k \geq 0$,

$$T_k(g)(z) := \sum_{i=0}^k \frac{g^{(i)}(0)}{i!} z^i,$$

where $g^{(i)}$ is the i th derivative of g . In other words, $T_k(g)(z)$ is the first $k + 1$ terms of the Taylor expansion of $g(z)$ at the origin. Then [3, Lemma 2.2.1] states the following.

PROPOSITION 7. *Let $P(z) = \sum_{i=0}^n c_i z^i$ be a polynomial such that for some $\beta > 1$, $P(z)$ is zero-free in the (closed) disk of radius β centered at the origin. Then there exists a constant C_β such that for any $0 < \varepsilon < 1$,*

$$\left| \frac{\exp(T_k(g)(1))}{P(1)} - 1 \right| \leq \varepsilon,$$

where $k = C_\beta \log \frac{n}{\varepsilon}$.

This result states that we can approximately evaluate $P(1)$ using the first $O(\log \frac{n}{\varepsilon})$ terms of the Taylor expansion of $\log P(x)$ at the origin, when the polynomial is zero-free in the disk of radius $\beta > 1$. If our polynomial $P_G(x)$ is zero-free in the δ -strip of $[0, 1]$, then we can apply a transformation [3, Lemma 2.2.3], to transform it into a polynomial that is zero-free in the disk of radius > 1 .

The following lemma describes the construction.

LEMMA 8. *Let $0 < \delta < 1$ be a constant and $\beta = 1 + \frac{\exp(-\frac{1}{\delta})}{2 - 2\exp(-\frac{1}{\delta})} > 1$. There exists a polynomial $\phi_\delta(z)$ of degree $\exp(O(\frac{1}{\delta}))$ such that*

- (1) $\phi_\delta(0) = 0$ and $\phi_\delta(1) = 1$;
- (2) for every $z \in \mathbb{C}$ with $|z| \leq \beta$, the value $\phi_\delta(z)$ is within the 2δ -strip of $[0, 1]$.

PROOF. The idea to construct the polynomial ϕ_δ is to start with the function $\log(z)$ (the principal branch of the logarithm) by noting that the logarithm function maps a circle centered at zero to an interval orthogonal to the real axis. We can then scale and shift the function to restrict the interval to some desired region. Finally, we construct the polynomial ϕ_δ to approximate it.

To this end, we let $h(z) := \delta \log \frac{1}{1 - \alpha z}$ where α is a parameter to be set. The condition $h(0) = 0$ is automatically satisfied. To satisfy $h(1) = 1$, we set $\alpha = 1 - \exp(-\frac{1}{\delta})$. Then $\beta = 1 + \frac{\exp(-\frac{1}{\delta})}{2 - 2\exp(-\frac{1}{\delta})} = \frac{1 + \alpha}{2\alpha}$. Note that $\beta < \frac{1}{\alpha}$, so $h(z)$ is well-defined over the disk of radius β centered at the origin. It is easy to verify that for every $z \in \mathbb{C}$ with $|z| \leq \beta$, it holds that

$$-\delta \log 2 \leq \Re h(z) \leq 1 + \delta \log 2,$$

and

$$|\Im h(z)| \leq \frac{\pi}{2} \cdot \delta.$$

We use a polynomial, namely, the Taylor expansion of $h(z)$ at the origin to approximate $h(z)$. For every $k \geq 0$, the first k terms of the Taylor expansion of h at the origin is

$$T_k(h)(z) = \delta \sum_{i=1}^k \frac{\alpha^i}{i} \cdot z^i.$$

Then for $m = \lceil \frac{\log(10(1+\alpha)) - \log(1-\alpha)}{\log 2 - \log(1+\alpha)} \rceil = \exp\left(O\left(\frac{1}{\delta}\right)\right)$, since $|z| \leq \beta$, we have

$$|h(z) - T_k(z)| = \left| \delta \sum_{i=m+1}^{\infty} \frac{\alpha^i}{i} \cdot z^i \right| \leq \frac{2\delta}{(1-\alpha)(m+1)} \left(\frac{1+\alpha}{2}\right)^{m+1} \leq \frac{\delta}{10}.$$

In particular, we have

$$|T_m(h)(1) - 1| = |T_m(h)(1) - h(1)| \leq \frac{\delta}{10}.$$

Finally, we define

$$\phi_\delta(z) = \frac{T_m(h)(z)}{T_m(h)(1)}$$

to force $\phi_\delta(1) = 1$. This finishes the construction. \square

Therefore, for a polynomial $P(z)$ that is zero-free in the δ -strip of $[0, 1]$, we can use Proposition 7 to approximately evaluate $P_\phi(z) := P(\phi_{\frac{\delta}{2}}(z))$, which is zero-free in the disk of radius β at the origin for the value β defined in Lemma 8. Note that $P(\phi_{\frac{\delta}{2}}(1)) = P(1)$.

PROPOSITION 9. *Let $P(z)$ be a polynomial of degree n such that for some $\delta > 0$, $P(z)$ is zero-free in the δ -strip of $[0, 1]$. Then there exists a constant C_δ such that for any $0 < \varepsilon < 1$,*

$$\left| \frac{\exp\left(T_k(\log P_\phi)(1)\right)}{P(1)} - 1 \right| \leq \varepsilon,$$

where $k = C_\delta \log \frac{n}{\varepsilon}$.

At last, we show the Taylor expansion $T_k(\log P_\phi)(1)$ can be computed efficiently from the coefficients of P .

PROPOSITION 10. *Let $P(z)$ be a polynomial of degree n such that for some constant $\delta > 0$, $P(z)$ is zero-free in the δ -strip of $[0, 1]$. For every $0 \leq k \leq n$, assume that we have oracle access to the first k coefficients of $P(z)$, we can compute*

$$T_k(\log P_\phi)(1)$$

in time $O(k^2)$.

Since the degree of $\phi_{\frac{\delta}{2}}(z)$ is $\exp\left(O\left(\frac{1}{\delta}\right)\right)$, we can write $P_\phi(z) = \sum_{i=1}^m c_i z^i$ where $m = n + C_\delta$ for some constant C_δ depending only on δ . It is easy to compute the coefficients c_k given the coefficients of $P(z)$ of degree at most k in $O(k)$ time. Let $g_\phi := \log P_\phi$, we now show how to compute $T_k(g_\phi)$ using $(c_i)_{i \leq k}$.

Let z_1, \dots, z_m be the zeros of a polynomial $P_\phi(z)$ and for $0 \leq k \leq m$, let $p_k := \sum_{i=1}^m z_i^{-k}$ be the k th inverse power sum of the zeros of $P_\phi(z)$.

Newton's identities state the relation between $(p_k)_k$ and the coefficients $(c_i)_i$.

PROPOSITION 11 (NEWTON'S IDENTITY). *For every $1 \leq k \leq m$, it holds that*

$$k \cdot c_k = - \sum_{i=0}^{k-1} c_i \cdot p_{k-i}.$$

Newton's identities essentially provide a way to compute all p_k consecutively using $(c_i)_i$, and vice versa. To be specific,

$$p_0 = m,$$

$$p_k = -c_0^{-1} \cdot \left(\sum_{i=1}^{k-1} p_i \cdot c_{k-i} + k \cdot c_k \right) \text{ for } 1 \leq k \leq m.$$

Therefore, it costs $O(k^2)$ time to compute p_k using above recurrence.

However, we can write $P_\phi(z) = c_m \prod_{i=1}^m (z - z_i)$. Recall that $g_\phi(z) = \log P_\phi(z) = \log c_m + \sum_{i=1}^m \log(z - z_i)$.

It is easy to calculate that for any $i \geq 1$,

$$g_\phi^{(i)}(0) = -(i-1)! \sum_{j=1}^m z_j^{-i} = -(i-1)! p_i.$$

Therefore,

$$T_k(g_\phi)(z) := \log c_0 - \sum_{i=1}^k \frac{p_i}{i} z^i. \quad (5)$$

This proves Proposition 10.

3.1 Computing the Inverse Power Sums

Given Proposition 7 and Equation (5), the main task then reduces to computing the first k inverse power sums $(p_i)_{i \leq k}$. We follow the method of Patel and Regts [31].

We need some notations first. Let \mathcal{G} be a family of all graphs, and \mathcal{G}_k be all graphs with at most k vertices. We call a function $g : \mathcal{G} \rightarrow \mathbb{C}$ a *graph invariant* if $g(G) = g(H)$ whenever $G \simeq H$. A *graph polynomial* is a graph invariant $Q : \mathcal{G} \rightarrow \mathbb{C}[z]$, where $\mathbb{C}[z]$ is the polynomial ring over \mathbb{C} . We call a graph invariant $g(\cdot)$ *additive* if for any two graphs G and H , it holds that $g(G \sqcup H) = g(G) + g(H)$, where $G \sqcup H$ is the graph consisting of disjoint copies of G and H . Similarly, we call it *multiplicative* if for every two graphs G and H , it holds that $g(G \sqcup H) = g(G) \cdot g(H)$. For graphs H and G , we use $\#\text{Ind}(H, G)$ to denote the number of *induced* subgraphs of G isomorphic to H . Then $\#\text{Ind}(H, \cdot)$ is a graph invariant for a fixed graph H . By convention let $\#\text{Ind}(\emptyset, G) = 1$ for any G .

Definition 12. Let $Q(G)(z) = \sum_{i=1}^{d(G)} a_i(G) z^i$ be a multiplicative graph polynomial of degree $d(G)$ such that $Q(G)(0) = 1$ for any G . We call $Q(\cdot)$ a *bounded induced graph counting polynomial (BIGCP)* if there are constants $\alpha, \beta \in \mathbb{N}$ such that the following holds:

- for every graph G , there exist $\lambda_{H,i} \in \mathbb{C}$ such that

$$a_i(G) = \sum_{H \in \mathcal{G}_{\alpha i}} \lambda_{H,i} \cdot \#\text{Ind}(H, G); \quad (6)$$

- for every $H \in \mathcal{G}_{\alpha i}$, $\lambda_{H,i}$ can be computed in time $\exp(\beta \cdot |V(H)|)$, where $V(H)$ is the set of vertices of H .

Patel and Regts [31, Theorem 3.2] have shown that the inverse power sums can be computed for BIGCP in single exponential time.

PROPOSITION 13. *Let $\Delta \in \mathbb{N}$, G be a graph with maximum degree Δ and $Q(G)(\cdot)$ be a BIGCP. There is a deterministic $\exp(Ck \ln \Delta)$ -time algorithm, which computes the inverse power sums $(p_i)_{i \leq k}$ of $Q(G)(\cdot)$, for some constant $C > 0$.*

To our need, we just need to verify that $P_G(\cdot)$ from (4) is a BIGCP, whenever $f_0 = 1$.

LEMMA 14. Let $G = (V, E)$ be a Δ -regular graph and $f = [f_0, f_1, \dots, f_\Delta]$ be a signature. If $f_0 = 1$, then the Holant polynomial $P_G(\cdot)$ is a BIGCP with $\alpha = 2$ and $\beta = C\Delta$ for some constant $C > 0$.

PROOF. Clearly $P_G(0) = Z_0(G) = f_0^{|V|} = 1$. We would like to define $\lambda_{H,i}$ so that for every $1 \leq i \leq n$,

$$Z_i(G) = \sum_{H \in \mathcal{G}_{2i}} \lambda_{H,i} \cdot \#\text{Ind}(H, G). \quad (7)$$

For any $\sigma \in \{0, 1\}^E$, let $G[\sigma]$ be the subgraph induced by the set of vertices with at least 1 adjacent edges under σ . Let S_i be the set of subgraphs induced by assignments of Hamming weight i , namely, $S_i := \{G[\sigma] : \sigma \in \{0, 1\}^E \text{ and } |\sigma| = i\}$. The equivalence relation of graph isomorphisms induces a partition of S_i . We choose one graph from each equivalence class and denote this family of graphs by \mathcal{H}_i . Therefore, for every two distinct graphs $H_1, H_2 \in \mathcal{H}_i$, they are not isomorphic. Moreover, as $G[\sigma]$ has at most $2i$ vertices, $\mathcal{H}_i \subseteq \mathcal{G}_{2i}$.

For every $H \in \mathcal{H}_i$, consider an assignment π of signatures, where $v \in V$ of degree $d \leq \Delta$ is assigned $[f_0, f_1, \dots, f_d]$, a truncated f . Let

$$\lambda_{H,i} := Z_i(H; \pi).$$

To verify Equation (7), we rewrite

$$\begin{aligned} Z_i(G) &= \sum_{\sigma \in \{0,1\}^E \text{ and } |\sigma|=i} \prod_{v \in V} f(\sigma|_{E(v)}) \\ &= \sum_{H \in \mathcal{G}_{2i}} \sum_{\substack{\sigma \in \{0,1\}^E \\ |\sigma|=i \text{ and } G[\sigma] \simeq H}} \prod_{v \in V} f(\sigma|_{E(v)}) \\ &= \sum_{H \in \mathcal{G}_{2i}} \sum_{\substack{G' \text{ is an induced subgraph of } G \\ G' \simeq H}} \sum_{\substack{\sigma \in \{0,1\}^E \\ |\sigma|=i \text{ and } G[\sigma]=G'}} \prod_{v \in V} f(\sigma|_{E(v)}) \\ &= \sum_{H \in \mathcal{G}_{2i}} \sum_{\substack{G' \text{ is an induced subgraph of } G \\ G' \simeq H}} Z_i(G'; \pi) \cdot f_0^{|V \setminus V(H)|} \\ &= \sum_{H \in \mathcal{G}_{2i}} Z_i(H; \pi) \cdot \#\text{Ind}(H, G), \end{aligned}$$

since $Z_i(G'; \pi) = Z_i(H; \pi)$ whenever $G' \simeq H$. Thus, Equation (7) holds.

Since $\mathcal{H}_i \subseteq \mathcal{G}_{2i}$, we have that $\alpha = 2$. Moreover, H contains at most $\Delta |V(H)|$ edges. As a consequence, $Z_i(H; \pi)$ can be computed in time $2^{O(\Delta |H|)}$. Thus, we can take $\beta = C\Delta$ for some constant $C > 0$. \square

Gathering what we have seen so far, we have the following theorem.

THEOREM 15. Let f be a symmetric signature of arity Δ . If the local polynomial $P_f(x)$ is H_ε -stable for some $\varepsilon > 0$, then there is an FPTAS for $\text{Holant}(f)$.

PROOF. Since $P_f(x)$ is H_ε -stable, $f_0 \neq 0$. We may thus normalize f so that $f_0 = 1$. By Lemma 6, $P_f(x)$ being H_ε -stable implies that for any Δ -regular $G = (V, E)$, $P_G(x)$ is zero-free in a δ -strip containing $[0, 1]$. Recall that $Z(G; f) = P_G(1)$. By Proposition 9, we can $(1 \pm \varepsilon)$ -approximate $P_G(1)$ using $\exp(T_k(\log P_G)(x))$ for some $k = O\left(\log \frac{m}{\varepsilon}\right)$, where $m = |E|$. To compute $T_k(\log P_G)(x)$, we use Proposition 13 and Lemma 14 to compute the inverse power sums (p_i) of $P_G(x)$, and then apply Proposition 11 to get the first k coefficients of $P_G(x)$. The theorem then follows from Proposition 10. \square

Remark. Theorem 15 is a sufficient but not necessary condition for a Holant problem to be approximable. To see this, once again, consider the problem of counting even subgraphs discussed near the end of Section 2.

4 HOLOGRAPHIC TRANSFORMATIONS

Theorem 15 implies an FPTAS for $\text{Holant}(f)$ if f is H_ε -stable. However, an FPTAS may still exist even if f is not H_ε -stable. One way to extend the reach of this approach is via Valiant's holographic transformation [42], which changes f but preserves the partition function. We remark that even with holographic transformations, this approach is not exhaustive. An example is the problem of counting even subgraphs.

We use $\text{Holant}(f \mid g)$ to denote the Holant problem where the input is a bipartite graph $H = (U, V, E)$. Each vertex in U or V is assigned the signature f or g , respectively. Call this assignment π , namely, $\pi(u) = f$ for any $u \in U$ and $\pi(v) = g$ for any $v \in V$. Recall Equation (1), and $Z(H; \pi)$ is the output of the computational problem $\text{Holant}(f \mid g)$. The signature f is considered as a row vector (or covariant tensor) of length $2^{\text{arity}(f)}$ (by listing its truth table), whereas the signature g is considered as a column vector (or contravariant tensor) of length $2^{\text{arity}(g)}$.

Let T be an invertible 2-by-2 matrix. Let $d_1 = \text{arity}(f)$ and $d_2 = \text{arity}(g)$. Define $f' = f \cdot T^{\otimes d_1}$ and $g' = (T^{-1})^{\otimes d_2} g$. Let π' be the assignment such that $\pi'(u) = f'$ for any $u \in U$ and $\pi'(v) = g'$ for any $v \in V$.

PROPOSITION 16 (VALIANT'S HOLANT THEOREM [42]). *If $T \in \mathbb{C}^{2 \times 2}$ is an invertible matrix, then for any bipartite graph H , $Z(H; \pi) = Z(H; \pi')$, where π' is defined above.*

Therefore, an invertible holographic transformation does not change the complexity of the Holant problem in the bipartite setting. For a (non-bipartite) Holant problem, we can always view the edge as a binary equality function $=_2$. Thus, $\text{Holant}(f)$ is the same as $\text{Holant}(f \mid =_2)$. Let $\mathbf{O}_2(\mathbb{C})$ be the set of 2-by-2 orthogonal matrices, namely, $\mathbf{O}_2(\mathbb{C}) = \{T \in \mathbb{C}^{2 \times 2} \mid TT^\top = I_2\}$. As orthogonal transformations preserve the binary equality, the following result will become handy in the standard setting.

PROPOSITION 17 ([9]). *If $T \in \mathbf{O}_2(\mathbb{C})$ is an orthogonal matrix, then for any d -regular graph G and a signature f of arity d , $Z(G; f) = Z(G; f \cdot T^{\otimes d})$.*

As a particular consequence of Proposition 17, under the transformation $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, the complexity of $\text{Holant}(f)$ is equivalent to $\text{Holant}(\bar{f})$ where $\bar{f} = [f_d, f_{d-1}, \dots, f_0]$. We will use this fact in the following without explicitly mentioning it.

5 SECOND-ORDER RECURRENCES

The aim of this section is to study the locations of zeros of local polynomials of signatures satisfying generalised second-order recurrences to apply Theorem 15. Specifically, we identify the family of signatures whose local polynomials are H_ε -stable for some $\varepsilon > 0$, under some suitable holographic transformations.

For a tuple of reals $(a, b, c) \neq (0, 0, 0)$, define

$$\mathcal{F}_{a,b,c} := \{[f_0, f_1, \dots, f_d] : af_k + bf_{k+1} + cf_{k+2} = 0, \forall 0 \leq k \leq d-2, \text{ and } f_k \geq 0, \forall 0 \leq k \leq d\}.$$

The family $\mathcal{F}_{a,b,c}$ consists of signatures with non-negative entries satisfying second-order linear recurrence relation parameterized by (a, b, c) . Whenever $\mathcal{F}_{a,b,c}$ appears, we always assume that $(a, b, c) \neq (0, 0, 0)$.

The following proposition states the general form of a function satisfying a generalised second-order recurrence.

PROPOSITION 18. Let $f = [f_0, \dots, f_d] \in \mathcal{F}_{a,b,c}$ be a signature and $c \neq 0$. There are two cases:

- if $b^2 \neq 4ac$, then for every $k \in \{0, 1, \dots, d\}$,

$$f_k = x\phi_1^k + y\phi_2^k,$$

where ϕ_1, ϕ_2 are the two roots of the polynomial $cz^2 + bz + a = 0$ and x, y are two constants independent of k and determined by f_0 and f_1 ;

- if $b^2 = 4ac$, then for every $k \in \{0, 1, \dots, d\}$,

$$f_k = x\phi^k + yk\phi^{k-1},$$

where ϕ is the unique root of the polynomial $cz^2 + bz + a = 0$ and x, y are two constants independent of k and determined by f_0 and f_1 . In case of $\phi = 0$, we follow the convention that $0 \cdot 0^{-1} = 0$.

In this section, we assume that all signatures (or their reversals) in consideration have nonzero leading term, i.e., $f_0 \neq 0$. By considering the reversal if necessary, this assumption covers all cases where $f_0 \neq 0$ or $f_d \neq 0$. We will discuss the case of $f_0 = f_d = 0$ in Section 6.

We will use $\mathcal{F}_{a,b,c}^*$ to denote the subset family of $\mathcal{F}_{a,b,c}$ with this additional property $f_0 > 0$. It turns out that the behaviour of signatures in $\mathcal{F}_{a,b,c}^*$ is closely related to the sign of the value $b^2 - 4ac$, namely, the discriminant of the characteristic polynomial $cz^2 + bz + a$. Therefore, our discussion is divided into three parts.

5.1 $b^2 - 4ac > 0$

In this case, the characteristic polynomial of signatures in $\mathcal{F}_{a,b,c}^*$ has two distinct real roots. We first single out a special case.

LEMMA 19. Let f be a symmetric signature of arity $d \geq 3$, where d is an odd integer, $f_i \geq 0$ for all $i = 0, 1, \dots, d$, and f is not identically zero. If there exist $p, q, s, t \in \mathbb{R}$ such that $p^2 + q^2 = s^2 + t^2$, $ps + qt < 0$, and $f = (p, q)^{\otimes d} + (s, t)^{\otimes d}$, then up to a non-zero (positive) scalar, f or \bar{f} is $[1, 0, \lambda^2, 0, \dots, \lambda^{d-1}, 0]$ for some $\lambda > 1$, where $\bar{f} := [f_d, f_{d-1}, \dots, f_0]$.

PROOF. Since $f = (p, q)^{\otimes d} + (s, t)^{\otimes d}$, we have $f_i = q^i p^{d-i} + t^i s^{d-i}$. We consider cases depending on the sign of qt .

First assume $qt \geq 0$. The fact $f_1 \geq 0$ yields

$$qp^{d-1} + ts^{d-1} \geq 0.$$

Since d is odd, then q and t must be both non-negative. Then $t = \sqrt{p^2 + q^2 - s^2} \geq 0$. It follows from $ps + qt < 0$ that $ps < 0$. We can assume without loss of generality that $p > 0$, $s < 0$ and $|p| \geq |s|$ (a consequence of $f_0 \geq 0$). To ease the presentation, let $s' = -s > 0$. Then

$$ps + qt < 0 \iff qt < ps' \iff q^2(p^2 + q^2 - s'^2) < p^2 s'^2 \iff |q| < |s'|.$$

We then consider the requirement $f_{d-1} \geq 0$. This is equivalent to

$$\begin{aligned} q^{d-1}p + t^{d-1}s &\geq 0 \iff q^{d-1}p \geq t^{d-1}s' \\ &\iff q^2 p^{\frac{2}{d-1}} \geq (p^2 + q^2 - s'^2) s'^{\frac{2}{d-1}} \\ &\iff q^2 (p^{\frac{2}{d-1}} - s'^{\frac{2}{d-1}}) \geq (p^2 - s'^2) s'^{\frac{2}{d-1}}. \end{aligned}$$

We apply $|q| < |s'|$ and obtain

$$\begin{aligned} (p^2 - s'^2)s'^{\frac{2}{d-1}} \leq s'^2(p^{\frac{2}{d-1}} - s'^{\frac{2}{d-1}}) &\iff \frac{p^2}{s'^2} - 1 \leq \frac{p^{\frac{2}{d-1}}}{s'^{\frac{2}{d-1}}} - 1 \\ &\iff |s| \geq |p|. \end{aligned}$$

Therefore, it must hold that $p = -s$, $q = t$, and we have $f = (p, q)^{\otimes d} + (-p, q)^{\otimes d}$. Moreover, $ps + qt < 0$ implies that $p > q$. If $q = t = 0$, then f is identically zero, a contradiction. Otherwise $q > 0$, and we can choose $\lambda = \frac{p}{q} > 1$ and \bar{f} is $[1, 0, \lambda^2, 0, \lambda^4, 0, \dots]$ up to a non-zero scalar.

Now, we assume $qt < 0$, and without loss of generality further assume that $q > 0$ and $t < 0$. Then $t = -\sqrt{p^2 + q^2 - s^2}$. We distinguish between $ps \geq 0$ and $ps < 0$.

- (i) If $ps \geq 0$, then the reasoning is the same as the case of $qt \geq 0$ above, after exchanging p and q , s and t , and reversing f .
- (ii) If $ps < 0$, then we first assume that $p < 0$ and $s > 0$. In this case, we let $p' = -p$ and $t' = -t$. Then $f_0, f_1, f_2 \geq 0$ implies

$$s^d \geq p'^d; \quad t's^{d-1} \leq qp'^{d-1}; \quad t'^2s^{d-2} \geq q^2p'^{d-2},$$

where p', q, t', s above are all positive. The first two imply that $t'p' \leq qs$, and the last two imply that $t'p' \geq qs$. Thus, $t'p' = qs$. This is further equivalent to $s^2q^2 = p^2(p^2 + q^2 - s^2)$, or $(p^2 + q^2)(p^2 - s^2) = 0$. It implies that either $p = q = 0$ or $p = -s$. In both cases, f is identically zero, a contradiction.

Finally, consider the case when $p > 0$ and $s < 0$. Then $f_0 = p^d + s^d \geq 0$ implies $|p| \geq |s|$. However, $f_d = q^d + t^d \geq 0$ is equivalent to $|q| \geq |t|$. However $p^2 + q^2 = s^2 + t^2$. Thus, we have $p = -s$ and $q = -t$. This means that f is identically zero, also a contradiction. \square

Let $=_d$ be the equality function of arity d , namely, the function $[1, 0, \dots, 0, 1]$. If $\beta > 1$, then the problem Holant($=_d | [\beta, 1, \beta]$) is to compute the partition function of *ferromagnetic Ising model without external fields*. An FPRAS for this problem has been given by Jerrum and Sinclair [22]. Then we have the following lemma.

LEMMA 20. *Let $f = [f_0, f_1, \dots, f_d] \in \mathcal{F}_{a,b,c}^*$ with $b^2 - 4ac > 0$. Then one of the following holds:*

- Holant(f) can be solved exactly in polynomial-time; or
- there is an invertible matrix $M \in \mathbb{C}^{2 \times 2}$ such that Holant($f \cdot M^{\otimes d} | (M^{-1})^{\otimes 2} \cdot (=2)$) is a ferromagnetic Ising model without external fields; or
- there is an orthogonal matrix $M \in \mathbf{O}_2(\mathbb{C})$ such that either $P_{f \cdot M^{\otimes d}}(z)$ or $P_{\bar{f} \cdot M^{\otimes d}}(z)$ is H_ε -stable for some $\varepsilon > 0$, where $\bar{f} := [f_d, f_{d-1}, \dots, f_0]$; or
- f or \bar{f} is $[1, 0, \lambda^2, 0, \lambda^4, 0, \dots, \lambda^{d-1}, 0]$ for some $\lambda > 1$ and has an odd arity d .

PROOF. If $c = 0$, then $b \neq 0$ and $af_k + bf_{k+1} = 0$ for all $k \leq d-2$. Thus, f_0, \dots, f_{d-1} form a geometric sequence with ratio $\phi = -\frac{a}{b} \in \mathbb{R}$, and f can be written as $f = x(1, \phi)^{\otimes d} + y(0, 1)^{\otimes d}$, where $x, y, \phi \in \mathbb{R}$. Pulling x and y into the tensor power, there exist $p, q, s, t \in \mathbb{R}$ and $r = 1$ or -1 such that f is a non-zero multiple of $(p, q)^{\otimes d} + r(s, t)^{\otimes d}$.

Otherwise, $c \neq 0$. It follows from Proposition 18 that we can rewrite $f = x(1, \phi_1)^{\otimes d} + y(1, \phi_2)^{\otimes d}$, where $\phi_1, \phi_2 \in \mathbb{R}$ and $\phi_1 \neq \phi_2$. Since f has non-negative weights, it implies that $x, y \in \mathbb{R}$ as well. Thus, similar to the case above, there exist $p, q, s, t \in \mathbb{R}$ and $r = 1$ or -1 such that f is a non-zero multiple of $(p, q)^{\otimes d} + r(s, t)^{\otimes d}$.

The four possibilities of the lemma come from the values p, q, s, t might take. If $pt = qs$, then f is degenerate and the partition function can be computed in polynomial time. This is because the

constraint factors into unary ones (see, e.g., Reference [6, Chapter 2]). Thus, we assume $pt - qs \neq 0$ in the following.

First, we consider the case that $p^2 + q^2 = s^2 + t^2$. We claim that we can always write $f = (p, q)^{\otimes d} + (s, t)^{\otimes d}$ without loss of generality. To see this, we distinguish between the parity of d . If d is odd, then $(p, q)^{\otimes d} - (s, t)^{\otimes d} = (p, q)^{\otimes d} + (-s, -t)^{\otimes d}$. If d is even, then we know from $f = (p, q)^{\otimes d} - (s, t)^{\otimes d}$ that $f_0 = p^d - s^d$ and $f_d = q^d - t^d$. Therefore, $f_0 > 0$ and $f_d \geq 0$ imply $p^2 > s^2$ and $q^2 \geq t^2$, which contradicts $p^2 + q^2 = s^2 + t^2$.

We write $\text{Holant}(f)$ as $\text{Holant}(f \mid =_2)$. Note that $M' = \begin{bmatrix} p & q \\ s & t \end{bmatrix}$ is an invertible matrix due to $pt - qs \neq 0$ and let $M = M'^{-1}$. It follows from Proposition 16 that $\text{Holant}(f \mid =_2)$ is equivalent to $\text{Holant}(f' \mid g')$, where $f' := f \cdot M^{\otimes d}$ and $g' := (M^{-1})^{\otimes 2} \cdot (=_2)$. We verify that this particular Holant problem is either solvable in polynomial-time or equivalent to a ferromagnetic Ising model without external fields. We have

$$f' = f \cdot M^{\otimes d} = \left((1, 0)^{\otimes d} + (0, 1)^{\otimes d} \right) M'^{\otimes d} M^{\otimes d} = (1, 0)^{\otimes d} + (0, 1)^{\otimes d},$$

and

$$g' = (M^{-1})^{\otimes 2} \cdot (=_2) = M'^{\otimes 2} \cdot (=_2) = (p^2 + q^2, ps + qt, ps + qt, s^2 + t^2)^{\top}.$$

If $ps + qt = 0$, then clearly $\text{Holant}(f' \mid g')$ is solvable in polynomial-time, since the edges in every component of the instance must be assigned with the same value to contribute a non-zero weight to the partition function. If $ps + qt > 0$, then we have that $(p^2 + q^2)(s^2 + t^2) - (ps + qt)^2 = (pt - qs)^2 > 0$, and $\text{Holant}(f' \mid g')$ is a ferromagnetic Ising model without external fields. If $ps + qt < 0$ and d is even, then a further transformation $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ makes the middle term positive, and it is a ferromagnetic Ising model again. Last, if $ps + qt < 0$ and d is odd, then Lemma 19 applies and we are in the last case of the lemma.

The remaining case is that $pt \neq qs$ and $p^2 + q^2 \neq s^2 + t^2$. If $|q| = |t|$, then $|p| \neq |s|$ and we replace (p, q, s, t) by (q, p, t, s) . This is equivalent to work with \bar{f} . So from now on, we also assume that $|q| \neq |t|$. Let $M' = \begin{bmatrix} w & 1 \\ 1 & -w \end{bmatrix}$ where $w \in \mathbb{R}$ is a parameter to be set later. Then $f \cdot M'^{\otimes d}$ is $(q + pw, p - qw)^{\otimes d} + r(t + sw, s - tw)^{\otimes d}$ and

$$P_{f \cdot M'^{\otimes d}}(z) = (q + pw + (p - qw)z)^d + r(t + sw + (s - tw)z)^d.$$

Recall that $r = 1$ or -1 , so the zeros of this polynomial must satisfy

$$|q + pw + (p - qw)z| = |t + sw + (s - tw)z|. \quad (8)$$

We show that by choosing appropriate w the roots to this equation are in the open left half-plane.

First consider $p = q = 0$. Since $p^2 + q^2 \neq s^2 + t^2$, it holds that $(s, t) \neq (0, 0)$. We will choose w so that $s - tw \neq 0$, in which case the root to Equation (8) must be $-\frac{t+sw}{s-tw}$. There are four cases.

- If $t = 0$, then let $w = 1$. It holds that $s - tw = s \neq 0$ and $-\frac{t+sw}{s-tw} = -w < 0$.
- If $s = 0$, then let $w = -1$. It holds that $s - tw = t \neq 0$ and $-\frac{t+sw}{s-tw} = \frac{1}{w} < 0$.
- If $st < 0$, then let $w = \frac{2s}{t} < 0$. It holds that $s - tw = -s \neq 0$ and $-\frac{t+sw}{s-tw} = \frac{t}{s} + w < 0$.
- If $st > 0$, then let $w = 0$. It holds that $s - tw = s \neq 0$ and $-\frac{t+sw}{s-tw} = -\frac{t}{s} < 0$.

The case of $s = t = 0$ is completely analogous.

Now we can make the further assumption that $(p, q) \neq (0, 0)$ and $(s, t) \neq (0, 0)$. Let $\alpha = -\frac{p-qw}{s-tw} \in \mathbb{R}$ be another parameter, which eventually will be set to 1 or -1 . As $w = \frac{\alpha s + p}{\alpha t + q}$ and $|q| \neq |t|$, the value of the parameter w will be determined when the sign of α is chosen. Since $p - qw = \frac{\alpha(pt - qs)}{\alpha t + q} \neq 0$,

we let $z_1 = -\frac{q+pw}{p-qw}$, which is well-defined. Similarly, it holds that $s - tw = \frac{qs-pt}{\alpha t+q} \neq 0$, and we let $z_2 = -\frac{t+sw}{s-tw}$. Equation (8) is equivalent to

$$|\alpha| \cdot |z - z_1| = |z - z_2|. \quad (9)$$

Since $|\alpha| = 1$, to make the roots to Equation (9) in the open left half-plane, it suffices to make sure that

$$z_1 + z_2 = \frac{(p^2 + q^2) - (s^2 + t^2)}{\alpha(qs - pt)} < 0. \quad (10)$$

Since $p^2 + q^2 \neq s^2 + t^2$, we can let $\alpha = -1$ if $\frac{(p^2+q^2)-(s^2+t^2)}{qs-pt} > 0$, or let $\alpha = 1$ otherwise.

We have showed that there is a matrix $M' \in \mathbb{C}^{2 \times 2}$ such that the zeros of $P_{f \cdot M'^{\otimes d}}(z)$ are in the open left half-plane. Since a polynomial has only a finite number of zeros, there is a constant $\varepsilon > 0$ that $P_{f \cdot M'^{\otimes d}}(z)$ is H_ε -stable. It holds that $M'(M')^\top = \begin{bmatrix} 1+w^2 & 0 \\ 0 & 1+w^2 \end{bmatrix} = (1+w^2)I_2$ where $1+w^2 > 0$ as $w \in \mathbb{R}$. Let $M = \frac{1}{\sqrt{1+w^2}}M'$. Clearly $MM^\top = I_2$ and $M \in \mathbf{O}_2(\mathbb{C})$. Since $P_{f \cdot M'^{\otimes d}}(z) = (1+w^2)^{d/2}P_{f \cdot M^{\otimes d}}(z)$, $P_{f \cdot M^{\otimes d}}(z)$ has the same set of zeros as $P_{f \cdot M'^{\otimes d}}(z)$. So $P_{f \cdot M^{\otimes d}}$ is also H_ε -stable for some $\varepsilon > 0$. \square

5.2 $b^2 - 4ac = 0$

When the characteristic polynomial of f has only one real root of multiplicity two, we show that there always exists an orthogonal transformation to reduce f to a function whose local polynomial is H_ε -stable.

LEMMA 21. *Let $f = [f_0, f_1, \dots, f_d] \in \mathcal{F}_{a,b,c}^*$ with $b^2 - 4ac = 0$, then there is an orthogonal matrix $M \in \mathbf{O}_2(\mathbb{C})$ such that $P_{f \cdot M^{\otimes d}}(z)$ is H_ε -stable for some $\varepsilon > 0$.*

PROOF. If $c = 0$, then $b = 0$, since $b^2 - 4ac = 0$. It implies that $f_0 = 0$, which contradicts the definition of $\mathcal{F}_{a,b,c}^*$. Thus, $c \neq 0$.

Assume $b = 0$. Then $a = 0$, since $c \neq 0$ and $b^2 - 4ac = 0$. In this case, f is of form $[f_0, f_1, 0, \dots, 0]$ and we can simply pick $M = I_2$. Clearly $P_{f \cdot M^{\otimes d}}(z) = f_0 + df_1z$, which is H_ε -stable for some $\varepsilon > 0$, since $f_0 > 0$ and $f_1 \geq 0$.

Now we assume that $b \neq 0$. Since $c \neq 0$ and $b^2 - 4ac = 0$, the equation $cz^2 + bz + a = 0$ has one real root with multiplicity two and we denote it by ϕ . Note that $\phi = -\frac{b}{2c} \neq 0$, since $b \neq 0$. It follows from Proposition 18 that $f_k = x\phi^k + y'k\phi^{k-1}$ for $0 \leq k \leq d$ and some $x, y' \in \mathbb{R}$. Since $\phi \neq 0$, to ease the presentation, we let $y = \frac{y'}{\phi}$ and rewrite $f_k = x\phi^k + yk\phi^k$. Clearly $x = f_0 > 0$. By comparing entries of every Hamming weight, one can verify the following form of f :

$$f = x(1, \phi)^{\otimes d} + y \sum_{k=1}^d (1, \phi)^{\otimes(k-1)} \otimes (0, \phi) \otimes (1, \phi)^{\otimes(d-k)}.$$

Let $M' = \begin{bmatrix} 1 & w \\ -w & 1 \end{bmatrix}$ where $w \in \mathbb{R}$ is a parameter to be set later. Then,

$$f \cdot M'^{\otimes d} = x(1 - \phi w, \phi + w)^{\otimes d} + y \sum_{k=1}^d (1 - \phi w, \phi + w)^{\otimes(k-1)} \otimes (-\phi w, \phi) \otimes (1 - \phi w, \phi + w)^{\otimes(d-k)},$$

and

$$P_{f \cdot M'^{\otimes d}}(z) = x(1 - \phi w + (\phi + w)z)^d + yd(1 - \phi w + (\phi + w)z)^{d-1}(-\phi w + \phi z).$$

The zeros of this polynomial must satisfy

$$(1 - \phi w + (\phi + w)z)^{d-1}(x - (x + yd)\phi w + (xw + (x + yd)\phi)z) = 0. \quad (11)$$

If $\phi + w \neq 0$ and $xw + (x + yd)\phi \neq 0$, then the roots of this equation are $-\frac{1-\phi w}{\phi+w}$ or $-\frac{x-(x+yd)\phi w}{xw+(x+yd)\phi}$. We choose appropriate w and check that these two roots are negative, $\phi + w \neq 0$ and $xw + (x + yd)\phi \neq 0$. Recall that $\phi \neq 0$ and $x = f_0 > 0$. We discuss various cases depending on the sign of ϕ and $x + yd$.

- If $x + yd = 0$, then the roots of Equation (11) are $-\frac{1-\phi w}{\phi+w}$ and $-\frac{1}{w}$. If $\phi < 0$, then let $w = -2\phi > 0$ and $-\frac{1-\phi w}{\phi+w} = \frac{1+2\phi^2}{\phi} < 0$. If $\phi > 0$, then let $w = \frac{1}{2\phi} > 0$ and $-\frac{1-\phi w}{\phi+w} = -\frac{1}{2\phi+\frac{1}{\phi}} < 0$. Clearly $\phi + w \neq 0$ and $xw + (x + yd)\phi \neq 0$ in both cases.
- If $\phi > 0$ and $x + yd > 0$, then let $w = \min\left\{\frac{1}{2\phi}, \frac{x}{2(x+yd)\phi}\right\} > 0$. It holds that

$$\begin{aligned} -\frac{1-\phi w}{\phi+w} &\leq -\frac{1}{2(\phi+w)} < 0, \\ \frac{x-(x+yd)\phi w}{xw+(x+yd)\phi} &\leq -\frac{x}{2(xw+(x+yd)\phi)} < 0. \end{aligned}$$

Whether $w = \frac{1}{2\phi}$ or $w = \frac{x}{2(x+yd)\phi}$, it is clear that $\phi + w \neq 0$ and $xw + (x + yd)\phi \neq 0$.

- If $\phi > 0$ and $x + yd < 0$, then $f_d = \phi^d(x + yd) < 0$. This contradicts to $f_d \geq 0$.
- If $\phi < 0$ and $x + yd > 0$, then consider $f_d = \phi^d(x + yd)$. If d is odd, then $f_d < 0$. Contradiction. Thus, d must be even. Then $\phi^{d-1} < 0$. Since $f_{d-1} = \phi^{d-1}(x + y(d-1)) \geq 0$, it holds that $x + y(d-1) \leq 0$. As $x > 0$, y must be negative, and then it contradicts to $x + yd > 0$.
- If $\phi < 0$ and $x + yd < 0$, then consider $f_d = \phi^d(x + yd)$. If d is even, then $f_d < 0$. But f_d must be non-negative, so d must be odd. Then $\phi^{d-1} > 0$. Since $f_{d-1} = \phi^{d-1}(x + y(d-1)) \geq 0$, it holds that $x + y(d-1) \geq 0$. Since $d > 1$, we can similarly deduce that $x + y(d-2) \leq 0$. This contradicts that $x > 0$ and $x + y(d-1) \geq 0$.

We have showed that there is a matrix $M' \in \mathbb{C}^{2 \times 2}$ such that the zeros of $P_{f \cdot M'^{\otimes d}}(z)$ are in the open left half-plane. Since a polynomial has only a finite number of zeros, there is a constant $\varepsilon > 0$ that $P_{f \cdot M'^{\otimes d}}(z)$ is H_ε -stable. It holds that $M'M'^T = \begin{bmatrix} 1+w^2 & 0 \\ 0 & 1+w^2 \end{bmatrix} = (1+w^2)I_2$ where $1+w^2 > 0$ as $w \in \mathbb{R}$. Let $M = \frac{1}{\sqrt{1+w^2}}M'$, and clearly $M \in \mathcal{O}_2(\mathbb{C})$. Since $P_{f \cdot M'^{\otimes d}}(z) = (1+w^2)^{d/2}P_{f \cdot M^{\otimes d}}(z)$, $P_{f \cdot M^{\otimes d}}(z)$ has the same set of zeros as $P_{f \cdot M'^{\otimes d}}(z)$. So $P_{f \cdot M^{\otimes d}}$ is also H_ε -stable for some $\varepsilon > 0$. \square

5.3 $b^2 - 4ac < 0$

When the characteristic polynomial of f has two distinct complex roots, we show that the local polynomial of f itself is H_ε -stable.

LEMMA 22. *Let $f = [f_0, f_1, \dots, f_d] \in \mathcal{F}_{a,b,c}^*$ with $b^2 - 4ac < 0$, then $P_f(z)$ is H_ε -stable for some $\varepsilon > 0$.*

PROOF. It holds that $c \neq 0$, since otherwise $b^2 - 4ac \geq 0$. Since $c \neq 0$ and $b^2 - 4ac < 0$, it follows from Proposition 18 that $f_k = x\phi^k + y\bar{\phi}^k$ for $0 \leq k \leq d$, where $\phi, \bar{\phi}$ are the two conjugate roots of the polynomial $cz^2 + bz + a = 0$ and $x, y \in \mathbb{C}$ are constants. Clearly, $x + y = f_0$ and $x\phi + y\bar{\phi} = f_1$. Since f_0 is real, it holds that $\Im(y) = -\Im(x)$. Since f_1 is real and $f_1 = x\phi + y\bar{\phi} = (x+y)\Re(\phi) + i(x-y)\Im(\phi)$, it holds that $\Re(x) = \Re(y)$. Thus, $y = \bar{x}$ and $f_k = x\phi^k + \bar{x}\bar{\phi}^k$ for $0 \leq k \leq d$. We write

$$f = x(1, \phi)^{\otimes d} + \bar{x}(1, \bar{\phi})^{\otimes d} \text{ and}$$

$$P_f(z) = x(1 + \phi z)^d + \bar{x}(1 + \bar{\phi} z)^d.$$

The zeros of $P_f(z)$ must satisfy

$$|x| \cdot |1 + \phi z|^d = |\bar{x}| \cdot |1 + \bar{\phi} z|^d. \quad (12)$$

Note that $\phi \neq 0$, and $x \neq 0$, since otherwise $\bar{x} = 0$ and f would be $[0, 0, \dots, 0]$. So, Equation (12) is equivalent to

$$\left| z - \left(-\frac{1}{\phi} \right) \right| = \left| z - \left(-\frac{1}{\bar{\phi}} \right) \right|.$$

Since $-\frac{1}{\phi}$ and $-\frac{1}{\bar{\phi}}$ are the complex conjugates of each other, the roots of this equation and thus the zeros of $P_f(z)$ must lie on the real axis. However, if $z \geq 0$, then

$$P_f(z) = \sum_{k=0}^d \binom{d}{k} f_k \cdot z^k > 0,$$

since $f_0 > 0$. Thus, the zeros of $P_f(z)$ are negative reals. Since a polynomial has only a finite number of zeros, there is a constant $\varepsilon > 0$ such that $P_f(x)$ is H_ε -stable. \square

6 EXCEPTIONAL CASES

Section 5 covered all signatures in $\mathcal{F}_{a,b,c}$ unless $f_0 = f_d = 0$. We discuss the remaining cases in this section. We will classify all of them, but the approximation complexity in one case (case (1) of Theorem 1) is still open.

Let $b \in \mathbb{R}$, and $b < 0$. Define \mathcal{A}_b to be the following class:

$$\left\{ [f_0, f_1, \dots, f_d] \mid \forall 0 \leq k \leq d-2, \frac{b^2}{4 \cos^2 \frac{\pi}{d}} f_k + b f_{k+1} + f_{k+2} = 0, f_0 = 0 \text{ and } f_1 > 0 \right\}.$$

Notice that \mathcal{A}_b is a special case of $\mathcal{F}_{a,b,c}$ except that the parameter a depends on the arity d . In fact, if $f \in \mathcal{A}_b$, then we can scale f so that f has the following form:

$$\left[0, \lambda \sin \frac{\pi}{d}, \lambda^2 \sin \frac{2\pi}{d}, \dots, \lambda^{d-1} \sin \frac{(d-1)\pi}{d}, 0 \right],$$

for $\lambda = -\frac{b}{2 \cos \frac{\pi}{d}} > 0$. (Recall that $b < 0$.) Namely, $f_i = \lambda^i \sin \frac{i\pi}{d}$.

LEMMA 23. *Let $f = [f_0, f_1, \dots, f_d] \in \mathcal{F}_{a,b,c}$ for some $d \geq 3$. If $f_0 = f_d = 0$, then there are three possibilities:*

- (I) $f \in \mathcal{A}_b$ for some $b < 0$;
- (II) $[f_0, f_1, \dots, f_d]$ is of form $[0, *, 0, 0, \dots, 0]$ or its reversal $[0, 0, \dots, 0, *, 0]$;
- (III) $[f_0, f_1, \dots, f_d]$ is of form $\lambda[0, 1, 0, \mu, 0, \mu^2, \dots, 0, \mu^{\frac{d-2}{2}}, 0]$ for some $\lambda, \mu > 0$ and even d .

PROOF. We start by considering the case $c = 0$. Then, $a f_k + b f_{k+1} = 0$ for every $0 \leq k \leq d-2$. It is easy to verify that f is identically 0 as $f_0 = f_d = 0$, which belongs to type II. Thus, we may assume that $c \neq 0$ and normalise c to 1 in the following. There are two further cases depending on whether $b^2 - 4a = 0$.

The first case is when $b^2 - 4a \neq 0$. It follows from Proposition 18 that $f_0 = x + y = 0$ and $f_d = x\phi_1^d + y\phi_2^d = 0$. These two identities together imply

$$x(\phi_1^d - \phi_2^d) = 0,$$

which further implies either $x = y = 0$ (and therefore $f_k = 0$ for all k) or $\phi_1^d = \phi_2^d$. We only need to discuss the case when $\phi_1^d = \phi_2^d$ and $x \neq 0$. There are two possibilities.

- (1) If $\frac{\phi_1}{\phi_2} \in \mathbb{R}$, then $\phi_1 = -\phi_2$ as $b^2 \neq 4a$. It implies that d is even. This is type III.
- (2) Otherwise, $\frac{\phi_1}{\phi_2} \notin \mathbb{R}$. In this case, $b^2 - 4a < 0$ and ϕ_1 and ϕ_2 are conjugate of each other. By swapping ϕ_1 and ϕ_2 if necessary, we may assume that $0 < \arg \phi_1 < \pi$. Then there exists some integer $0 < t < d$ so that $\arg \phi_1 = \frac{t\pi}{d}$ and $\frac{\phi_1}{\phi_2} = e^{\frac{2t\pi}{d}i} \notin \mathbb{R}$. Furthermore, $t \neq d/2$ as otherwise $\frac{\phi_1}{\phi_2} \in \mathbb{R}$. Since $a > b^2/4 \geq 0$, $|\phi_1| = |\phi_2| = \sqrt{a}$, and

$$f_k = x(\phi_1^k - \phi_2^k) = 2x \cdot a^{\frac{k}{2}} \left(\sin \frac{tk\pi}{d} \right) i.$$

Recall that we have the further requirement $f_k \geq 0$ for every $0 \leq k \leq d$. For $k = 1$, as $0 < t < d$, $\sin \frac{t\pi}{d} > 0$, and thus x must lie on the negative imaginary axis. Then, it must be that $\sin \frac{tk\pi}{d} \geq 0$ for all $0 \leq k \leq d$. If $t > 1$, then taking $k = \lfloor \frac{d}{t} \rfloor + 1 \leq d$ implies a contradiction. Thus, $t = 1$.

The assumption $0 < \arg \phi_1 < \pi$ implies that $\cos \frac{\pi}{d} = \frac{-b}{2\sqrt{a}} > 0$. Thus, $b < 0$ and $a = \frac{b^2}{4 \cos^2 \frac{\pi}{d}}$. This verifies that f is of type I.

At last we turn to the case that $b^2 - 4a = 0$. It follows from Proposition 18 that $f_k = x\phi^k + yk\phi^{k-1}$ where $\phi = -b/2$. Then $f_0 = 0$ means that $x = 0$, and $f_d = 0$ means that $y\phi^{d-1} = 0$. Thus, either $y = 0$ or $\phi = 0$, and any of the two cases implies that f is of type II. \square

Next, we show that type II and type III signatures are equivalent to approximately counting perfect matchings in general graphs. Denote by EXACTONE_d the function $[0, 1, 0, \dots, 0]$ of arity d , and by \mathcal{EO} the (infinite) set $\{\text{EXACTONE}_d \mid d \in \mathbb{N}^+\}$. Then $\text{Holant}(\mathcal{EO})$ is the problem of counting perfect matchings in a graph, denoted $\#PM$. (There is only one function for each degree/arity. So the mapping from vertices to functions is obvious for the infinite set \mathcal{EO} .)

For type III signatures, since multiplying by a constant does not change the complexity, we may assume that $\lambda = \sqrt{\mu}$. Then, $f = [0, \lambda, 0, \lambda^3, 0, \dots, \lambda^{d-1}, 0]$ with $\lambda > 0$. We will assume $\lambda < 1$. This is because that if $\lambda = 1$, then the problem is tractable exactly,¹ and if $\lambda > 1$, then taking its reversal makes $\lambda < 1$. We adopt the approximation-preserving reduction \leq_{AP} from Reference [13]. We often construct a gadget to express one constraint function via other functions. We call these gadget reductions and denote them by \leq_G . These reductions are parsimonious and are special cases of \leq_{AP} .

LEMMA 24. *Let $d \geq 4$ be an even integer, and $0 < \lambda < 1$. If $f = [0, \lambda, 0, \lambda^3, 0, \dots, \lambda^{d-1}, 0]$ of arity d , then*

$$\text{Holant}(\text{EXACTONE}_4) \leq_{AP} \text{Holant}(f).$$

PROOF. Applying a holographic transformation by $T = \begin{bmatrix} 1 & 0 \\ 0 & \lambda \end{bmatrix}$, we have that

$$\begin{aligned} \text{Holant}(f) &\equiv \text{Holant}\left(f \cdot (T^{-1})^{\otimes d} \mid T^{\otimes 2} \cdot \equiv_2\right) \\ &\equiv \text{Holant}([0, 1, 0, 1, 0, \dots, 1, 0] \mid [1, 0, \mu]), \end{aligned}$$

¹This falls into Reference [8, case 3 of Theorem 31]. More straightforwardly, the problem is to count the number of subgraphs where all vertices have an odd degree. For a connected graph G , the answer is $2^{|E|-|V|+1}$ if G has an even number of vertices, and 0 otherwise.

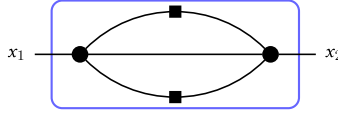


Fig. 2. A gadget for type III signatures, where squares are $[0, 1, 0]$, and circles are $[0, 1, 0, 1, 0]$. All edges, which can be viewed as degree two nodes on the right side in the bipartite formulation, have signatures $[1, 0, \mu]$. They are not drawn explicitly to avoid clutter.

where $0 < \mu = \lambda^2 < 1$. Thus, $\text{Holant}(f)$ is to count the number of odd subgraphs with edge weight μ in a d -regular graph. Here we adopted the bipartite Holant formulation to facilitate the holographic transformation. We will refer to vertices, which have signatures $[0, 1, 0, 1, 0, \dots, 1, 0]$, as the left side, and to edges, which have signatures $[1, 0, \mu]$ and are viewed as degree 2 nodes, as the right side.

We may add a self-loop to a vertex on the left by connecting it to an “edge” node on the right via two parallel edges. This operation simply reduces the degree of the vertex by 2, while leaving the constraint on the vertex unchanged, namely, that the vertex still must have an odd degree in the subgraph. Thus, with sufficiently many self-loops, we may simulate a binary disequality $[0, 1, 0]$ as well as an arity-4 signature $[0, 1, 0, 1, 0]$ on the left side of the bipartite Holant formulation.

Consider the gadget in Figure 2. We can use it effectively as a binary function with inputs (x_1, x_2) . Straightforward calculation yields its signature $(2\mu^2 + 2\mu^3)[1, 0, 1]$, and it is on the left (the two dangling edges do not have weights). Finally, with $[1, 0, 1]$ on the left, we can form a path of length n , and the resulting binary function is $[1, 0, \mu^n]$ on the right. More formally, we have the following chain of reductions:

$$\begin{aligned} \text{Holant}([0, 1, 0, 1, 0, \dots, 1, 0] \mid [1, 0, \mu]) &\geq_G \text{Holant}([0, 1, 0], [0, 1, 0, 1, 0] \mid [1, 0, \mu]) \\ &\geq_G \text{Holant}([1, 0, 1], [0, 1, 0, 1, 0] \mid [1, 0, \mu]) \\ &\geq_G \text{Holant}([0, 1, 0, 1, 0] \mid [1, 0, \mu^n]). \end{aligned}$$

The last problem is counting odd subgraphs with μ^n edge weights in 4-regular graphs and $\mu < 1$. Notice that the size of the construction scales linearly in n but the weight scales exponentially in n . As the edges have exponentially small weights, the contribution from all odd subgraphs is dominated by the contribution from minimum odd subgraphs, and minimum odd subgraphs are exactly perfect matchings. Thus, we can approximate the number of perfect matchings in a 4-regular graph $G = (V, E)$ if we could approximate $\text{Holant}([0, 1, 0, 1, 0] \mid [1, 0, \mu^n])$ on the same graph, where we choose n so that $\mu^n \leq 2^{-|E|}$. This finished the reduction. \square

Similar ideas can also handle the last case in Lemma 20, after taking its reversal and renaming λ .

LEMMA 25. *Let $d \geq 3$ be an odd integer, and $0 < \lambda < 1$. If $f = [0, \lambda, 0, \lambda^3, 0, \dots, \lambda^d]$ of arity d , then*

$$\text{Holant}(\text{EXACTONE}_3) \leq_{\text{AP}} \text{Holant}(f).$$

PROOF. As in the proof of Lemma 24, we do the same holographic transformation by $T = \begin{bmatrix} 1 & 0 \\ 0 & \lambda \end{bmatrix}$:

$$\text{Holant}(f) \equiv \text{Holant}([0, 1, 0, 1, 0, \dots, 1] \mid [1, 0, \mu]),$$

where $0 < \mu = \lambda^2 < 1$. Once again, with sufficiently many self-loops, we get $[0, 1, 0, 1]$ and $[0, 1]$ on the left side. Connecting $[0, 1]$ back to $[0, 1, 0, 1]$ through $[1, 0, \mu]$ yields $\mu[1, 0, 1]$ on the left. Thus, similar to the proof of Lemma 24, we can simulate $[1, 0, \mu^n]$ on the right. More formally, we

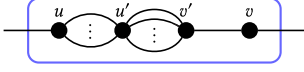


Fig. 3. A gadget for weighted equalities. There are n_1 edges between u and u' , and n_2 edges between u' and v' .

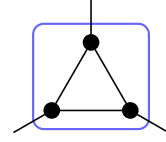


Fig. 4. A gadget to create $[0, 1, 0, 1]$.

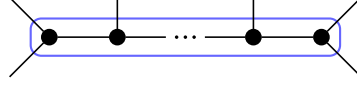


Fig. 5. A gadget to create $[1, 0, 1, 0, \dots, 1]$ or $[0, 1, 0, 1, \dots, 0]$.

have the following chain of reductions:

$$\begin{aligned}
 \text{Holant}([0, 1, 0, 1, 0, \dots, 1] \mid [1, 0, \mu]) &\geq_{\text{G}} \text{Holant}([0, 1], [0, 1, 0, 1] \mid [1, 0, \mu]) \\
 &\geq_{\text{G}} \text{Holant}([1, 0, 1], [0, 1, 0, 1] \mid [1, 0, \mu]) \\
 &\geq_{\text{G}} \text{Holant}([0, 1, 0, 1] \mid [1, 0, \mu^n]) \\
 &\geq_{\text{AP}} \text{Holant}(\text{EXACTONE}_3). \quad \square
 \end{aligned}$$

However, we have the following lemma.

LEMMA 26. *Let $d \geq 3$ be an integer and $0 < \lambda < 1$. Let $f = [0, \lambda, 0, \lambda^3, 0, \dots]$ be a symmetric signature of arity d . Then,*

$$\text{Holant}(f) \leq_{\text{AP}} \#\text{PM}.$$

PROOF. First, by the same holographic transformations as in the proofs of Lemmas 24 and 25,

$$\text{Holant}(f) \equiv \text{Holant}([0, 1, 0, 1, 0, \dots] \mid [1, 0, \mu]),$$

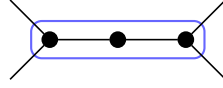
where $\mu = \lambda^2 > 0$.

Consider the gadget in Figure 3, where all vertices are the “exact one” function, namely, $[0, 1, 0, \dots, 0]$. It is easy to see that this gadget is equivalent to a weighted equality $[1, 0, \frac{n_2}{n_1}]$. Thus, we can use it to arbitrarily closely approximate $[1, 0, \mu]$ by tuning the integers n_1 and n_2 for any $\mu > 0$. To be more precise, suppose that we want approximation error ε . Then, we need to find n_1 and n_2 such that $\mu \leq \frac{n_2}{n_1} \leq \mu \left(1 + \frac{\varepsilon}{2k}\right)$ where k is the number of occurrences of $[1, 0, \mu]$ in the instance. If so, then we replace every occurrence of $[1, 0, \mu]$ by the gadget and the multiplicative error is at most $\left(1 + \frac{\varepsilon}{2k}\right)^k < 1 + \varepsilon$. To find n_1 and n_2 and bound their sizes, if μ is rational, then we can find constants n_1 and n_2 such that $\frac{n_1}{n_2} = \mu$. If μ is not rational, then it will depend on the model of computation. For finite precision models, we need to query and truncate μ up to $\Omega(\varepsilon/k)$ precision, and the resulting n_1 and n_2 are bounded linearly in k/ε .

In addition, consider the gadget in Figure 4, where, once again, all vertices are $[0, 1, 0, 0]$. The resulting signature is $[0, 1, 0, 1]$.

A simple calculation verifies that a sequence of d signatures $[0, 1, 0, 1]$ connected together, as in Figure 5, yields a signature $[0, 1, 0, \dots, 1, 0]$ of arity $d + 2$ if d is odd, or a signature $[1, 0, 1, 0, \dots, 1]$ of arity $d + 2$ if d is even. In the even case, to get $[0, 1, 0, 1, \dots, 0]$, we simply connect one of its dangling edges with $[0, 1, 0]$. Formally, we have the following sequence of reductions:

$$\begin{aligned}
 \text{Holant}([0, 1, 0, 1, 0, \dots] \mid [1, 0, \mu]) &\leq_{\text{G}} \text{Holant}(\{[0, 1, 0, 1, 0, \dots], [1, 0, \mu]\}) \\
 &\leq_{\text{AP}} \#\text{PM}. \quad \square
 \end{aligned}$$

Fig. 6. A gadget to create EXACTONE_d .

Lemmas 24, 25, and 26 together imply the following:

$$\text{Holant}(\text{EXACTONE}_4) \leq_{\text{AP}} \text{Holant}(f) \leq_{\text{AP}} \#\text{PM}, \text{ if } d \text{ is even,} \quad (13)$$

$$\text{Holant}(\text{EXACTONE}_3) \leq_{\text{AP}} \text{Holant}(f) \leq_{\text{AP}} \#\text{PM}, \text{ if } d \text{ is odd,} \quad (14)$$

where $f = [0, \lambda, 0, \lambda^3, 0, \dots]$ for some $0 < \lambda < 1$ has arity $d \geq 3$. Note that $\text{Holant}(\text{EXACTONE}_3)$ or $\text{Holant}(\text{EXACTONE}_4)$ is just an alias of counting perfect matchings in 3- or 4-regular graphs, which is equivalent to $\#\text{PM}$ in approximation. This is a folklore fact, and is shown in the next couple of lemmas.

LEMMA 27. $\text{Holant}(\text{EXACTONE}_3) \leq_{\text{AP}} \text{Holant}(\text{EXACTONE}_4)$.

PROOF. Note that a self-loop on $[0, 1, 0, 0, 0]$ gives $[0, 1, 0]$, and connecting it back to $[0, 1, 0, 0, 0]$ yields $[1, 0, 0]$. Thus,

$$\text{Holant}([0, 1, 0, 0, 0], [1, 0, 0]) \leq_{\text{G}} \text{Holant}([0, 1, 0, 0, 0]).$$

Given an instance G (namely, a 3-regular graph) of $\text{Holant}([0, 1, 0, 0])$, consider a disjoint union of G and its copy G' . We add a new vertex u for each pair v and v' , and connect u to both v and v' . Now all original vertices in G and G' have degrees exactly 4. Put $[0, 1, 0, 0, 0]$ on all these vertices, and $[1, 0, 0]$ on the newly introduced degree 2 vertices. It is easy to see that the partition function of this new instance is the square of the number of perfect matchings of G . Thus, we have the following reduction chain:

$$\begin{aligned} \text{Holant}([0, 1, 0, 0]) &\leq_{\text{AP}} \text{Holant}([0, 1, 0, 0, 0], [1, 0, 0]) \\ &\leq_{\text{G}} \text{Holant}([0, 1, 0, 0, 0]). \end{aligned} \quad \square$$

However, approximate counting perfect matchings in 3-regular graphs is as hard as that in general graphs.

LEMMA 28. $\#\text{PM} \leq_{\text{AP}} \text{Holant}(\text{EXACTONE}_3)$.

PROOF. Consider the gadget in Figure 6.

Notice that if we put $[0, 1, 0, 0]$ on the two degree three vertices, and $[0, 1, 0]$ on the middle vertex, the resulting signature is $[0, 1, 0, 0, 0]$. More generally, if we replace one of the degree three vertex by EXACTONE_d , then the resulting signature is EXACTONE_{d+1} . Namely, using this gadget, we can simulate the whole set of \mathcal{EO} , and

$$\#\text{PM} \leq_{\text{G}} \text{Holant}([0, 1, 0, 0], [0, 1, 0]).$$

Moreover, a self-loop on $[0, 1, 0, 0]$ gives $[0, 1]$, and connecting back to it gives $[1, 0, 0]$. By using the same squaring trick in Lemma 27, we can use $[1, 0, 0]$ as $[1, 0]$. Thus, we have the following reduction chain:

$$\begin{aligned} \#\text{PM} &\leq_{\text{G}} \text{Holant}([0, 1, 0, 0], [0, 1, 0]) \\ &\leq_{\text{G}} \text{Holant}([0, 1, 0, 0], [1, 0]) \\ &\leq_{\text{AP}} \text{Holant}([0, 1, 0, 0], [1, 0, 0]) \\ &\leq_{\text{G}} \text{Holant}([0, 1, 0, 0]). \end{aligned} \quad \square$$

Holant problems defined by type II signatures are counting perfect matchings in d -regular graphs. Clearly, by doing sufficiently many self-loops, either $\text{Holant}(\text{EXACTONE}_3) \leq_{\text{AP}} \text{Holant}(\text{EXACTONE}_d)$ or $\text{Holant}(\text{EXACTONE}_4) \leq_{\text{AP}} \text{Holant}(\text{EXACTONE}_d)$, depending on the parity of d . Thus, combining this fact with Lemma 27, Lemma 28, (13) and (14), we have the following result:

LEMMA 29. *Let $f = [0, 1, 0, \lambda^2, 0, \dots]$ for some $0 \leq \lambda < 1$. Then,*

$$\text{Holant}(f) \equiv_{\text{AP}} \# \text{PM}.$$

Notice that in Lemma 29, we factored out λ from the expression of f in Equations (13) and (14) to cover type II of Lemma 23 ($\lambda = 0$). Lemma 29 also covers type III of Lemma 23 and the last case in Lemma 20.

7 PROOF OF MAIN THEOREMS

We are now ready to assemble all the ingredients to prove our main theorems. We restate Theorem 1 for convenience.

THEOREM 1. *Let $f = [f_0, f_1, \dots, f_d]$ be a symmetric constraint function of arity $d \geq 3$ satisfying generalised second-order recurrences, and $f_i \geq 0$ for all $0 \leq i \leq d$. There is a fully polynomial-time (deterministic or randomised) approximation algorithm for $\text{Holant}(f)$, unless, up to a non-zero factor, f or its reversal is in one of the following form:*

- $[0, \lambda \sin \frac{\pi}{d}, \lambda^2 \sin \frac{2\pi}{d}, \dots, \lambda^i \sin \frac{i\pi}{d}, \dots, 0]$ for some $\lambda > 0$;
- $[0, 1, 0, \lambda, 0, \dots, 0, \lambda^{\frac{d-2}{2}}, 0]$ if d is even, or $[0, 1, 0, \lambda, 0, \dots, 0, \lambda^{\frac{d-1}{2}}]$ if d is odd, for some $0 \leq \lambda < 1$.

Moreover, in the latter case, approximating $\text{Holant}(f)$ is equivalent to approximately counting perfect matchings in general graphs.

PROOF. We apply Lemmas 20, 21, and 22. Then one of the following must happen:

- (1) $f_0 = f_d = 0$; or
- (2) f or \bar{f} is $[1, 0, \lambda^2, 0, \lambda^4, 0, \dots]$ for some $\lambda > 1$ and has an odd arity; or
- (3) $\text{Holant}(f)$ can be solved exactly in polynomial-time; or
- (4) there is an invertible matrix $M \in \mathbb{C}^{2 \times 2}$ such that $\text{Holant}(f \cdot M^{\otimes d} \mid (M^{-1})^{\otimes 2} \cdot (=)_2)$ is a ferromagnetic two-spin system; or
- (5) there is an orthogonal matrix $M \in \mathbf{O}_2(\mathbb{C})$ such that either $P_{f \cdot M^{\otimes d}}(z)$ or $P_{\bar{f} \cdot M^{\otimes d}}(z)$ is H_ε -stable for some $\varepsilon > 0$, where \bar{f} is the reversal of f .

We are done in Case (3), as well as in Case (5) by Proposition 17 and Theorem 15. In Case (4), we invoke the FPRAS by Jerrum and Sinclair [22]. In Cases (1) and (2), we are in the desired form of the theorem by Lemma 23. (In case $\mu > 1$ in Lemma 23, we can take its reversal so that $\mu < 1$, and if $\mu = 1$, then exact counting is tractable [8].) Finally, the approximation complexity of $[0, 1, 0, \lambda, 0, \lambda^2, 0, \dots]$ signatures is handled in Lemma 29. \square

Remark. It is worth noting that our algorithm applies beyond regular graphs. In fact, for any finite family of signatures \mathcal{F} , we can define $\text{Holant}(\mathcal{F})$ as the problem of computing the partition function on a graph G , where each vertex v of G is associated with a function $f_v \in \mathcal{F}$. It is straightforward to adapt the algorithm described in the proof of Theorem 1 for one to solve

Holant (\mathcal{F}).² It is not hard to see the adaptation provides an efficient approximation algorithm for Holant (\mathcal{F}) as long as there exists an orthogonal matrix $M \in \mathbf{O}_2(\mathbb{C})$ and $\varepsilon > 0$ such that $P_{f \cdot M^{\otimes d}}$ is H_ε -stable for every $f \in \mathcal{F}$, where d is the arity of f . Note that here M needs to work for the whole family \mathcal{F} and cannot depend on local signatures.

For example, we can let \mathcal{F} be the family of signatures for matchings up to arity d , or the family of signatures for edge covers up to arity d . In both cases one can simply take M to be the identity matrix and verify the condition above. Our algorithm thus recovers previously known deterministic approximation algorithms of counting matchings [4] and counting edge covers in bounded degree graphs [26] with a different approach. Notice that the finiteness of \mathcal{F} imposes a degree bound on the input graphs.

However, even for the same tuple (a, b, c) , signatures in $\mathcal{F}_{a,b,c}$ may require different M to be H_ε -stable. It is not clear how to obtain an algorithm in such cases.

We deduce Theorem 2 from Theorem 1 by noting that all ternary signatures satisfy generalised second-order recurrence relations. Therefore, we only need to deal with the case where $f = [0, a, b, 0]$ for some $a, b > 0$. We design an FPRAS for Holant (f) using the machinery called “winding” developed in References [20, 30]. The construction is sketched below without getting into too much technical details, which is out of the scope of the current article.

7.1 Windable and Strictly Terraced Functions

The FPRAS is obtained via MCMC approach, namely, we design a Markov chain to sample from certain distribution induced by Holant (f). Instead of directly doing so, we break every edge into two half edges and then sample from the state space consisting of all consistent edge assignment and assignments with *at most two* inconsistencies. It has been shown by McQuillan [30] that the Markov chain mixes rapidly as long as the signature f is *windable*. The following definition of windable functions is from Reference [20].

Definition 30. For any finite set J and any configuration $\mathbf{x} \in \{0, 1\}^J$, define $\mathcal{M}_{\mathbf{x}}$ as the set of partitions of $\{i \mid x_i = 1\}$ into pairs and at most one singleton. A function $f : \{0, 1\}^J \rightarrow \mathbb{R}_{\geq 0}$ is *windable* if there exist values $B(\mathbf{x}, \mathbf{y}, M) \geq 0$ for every $\mathbf{x}, \mathbf{y} \in \{0, 1\}^J$ and all $M \in \mathcal{M}_{\mathbf{x} \oplus \mathbf{y}}$, satisfying

- $f(\mathbf{x})f(\mathbf{y}) = \sum_{M \in \mathcal{M}_{\mathbf{x} \oplus \mathbf{y}}} B(\mathbf{x}, \mathbf{y}, M)$ for all $\mathbf{x}, \mathbf{y} \in \{0, 1\}^J$, and
- $B(\mathbf{x}, \mathbf{y}, M) = B(\mathbf{x} \oplus S, \mathbf{y} \oplus S, M)$ for all $\mathbf{x}, \mathbf{y} \in \{0, 1\}^J$ and all $S \in M \in \mathcal{M}_{\mathbf{x} \oplus \mathbf{y}}$.

Here $\mathbf{x} \oplus S$ denotes the vector obtained by changing x_i to $1 - x_i$ for the one or two elements i in S .

The definition of windable functions is technically involved. One can use the linear algebraic characterization for symmetric functions developed in Reference [20] to verify that $[0, a, b, 0]$ with nonnegative a and b is indeed windable:

THEOREM 31 ([20]). *A symmetric function $f : \{0, 1\}^3 \rightarrow \mathbb{R}_{\geq 0}$ is windable if and only if for every pinning g of f with arity $m \leq 3$, the function $h(x) = [h_0, h_1, \dots, h_m] := g(x)g(\bar{x})$ satisfies the following condition: The linear equation $\mathbf{A}_m \mathbf{x} = \mathbf{h}$ has a nonnegative solution $\mathbf{x} \geq 0$ where $\mathbf{h} = [h_0, h_1, \dots, h_{\lfloor \frac{m}{2} \rfloor}]$ and*

$$\mathbf{A}_1 = [1], \quad \mathbf{A}_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \mathbf{A}_3 = \begin{bmatrix} 3 & 0 \\ 1 & 2 \end{bmatrix}.$$

²The main adaptation is to show that $Z_i(G)$ is still a BIGCP when more than one constraint function are present. Since \mathcal{F} is finite, we can therefore view functions in \mathcal{F} as colors and enumerate *vertex colored induced subgraphs* instead of ordinary induced subgraphs in the proof of Lemma 14. A similar technique already appears in Reference [31].

However, to obtain an FPRAS for Holant(f), one requires the ratio between the total weight of *nearly consistent assignments* and that of consistent assignments is bounded by a polynomial in the size of the instance. This property is captured by the notion of “strictly terraced” defined in Reference [30].

Definition 32. A function $f : \{0, 1\}^J \rightarrow \mathbb{R}_{\geq 0}$ is *strictly terraced* if for every $\mathbf{x} \in \{0, 1\}^J$ and all $i, j \in J$:

$$f(\mathbf{x}) = 0 \Rightarrow f(\mathbf{x} \oplus \mathbf{e}_i) = f(\mathbf{x} \oplus \mathbf{e}_j),$$

where \mathbf{e}_i is the i th standard basis vector.

It is clear that $[0, a, b, 0]$ is strictly terraced when a, b are nonzero. In Reference [30], it is shown that the bounded ratio property holds for Holant instances with strictly terraced constraints. Therefore, we obtain an efficient Gibbs sampler for Holant(f), which can be turned into an FPRAS to compute the partition function via self-reduction [24]. Note that we only need self-reducibility for windable functions, which is straightforward to verify by Theorem 31.

Remark. The remaining open case in Theorem 1 is when $f \in \mathcal{A}_b$. Numerical evidence suggests that these functions are windable, via the criteria in Reference [20]. We conjecture that this is indeed the case, which would imply FPRAS for computing the partition functions of type I signatures, since these functions are strictly terraced as well.

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REFERENCES

- [1] Taro Asano. 1970. Theorems on the partition functions of the Heisenberg ferromagnets. *J. Phys. Soc. Japan* 29 (1970), 350–359.
- [2] Miriam Backens. 2018. A complete dichotomy for complex-valued Holant^c. In *Proceedings of the ICALP (LIPIcs)*, Vol. 107. 12:1–12:14.
- [3] Alexander I. Barvinok. 2016. *Combinatorics and Complexity of Partition Functions*. Algorithms and Combinatorics, Vol. 30. Springer.
- [4] Mohsen Bayati, David Gamarnik, Dimitriy A. Katz, Chandra Nair, and Prasad Tetali. 2007. Simple deterministic approximation algorithms for counting matchings. In *Proceedings of the STOC*. 122–127.
- [5] Andrei A. Bulatov. 2013. The complexity of the counting constraint satisfaction problem. *J. ACM* 60, 5 (2013), 34:1–34:41.
- [6] Jin-Yi Cai and Xi Chen. 2017. *Complexity Dichotomies for Counting Problems: Volume 1, Boolean Domain*. Cambridge University Press.
- [7] Jin-Yi Cai and Xi Chen. 2017. Complexity of counting CSP with complex weights. *J. ACM* 64, 3 (2017), 19:1–19:39.
- [8] Jin-Yi Cai, Heng Guo, and Tyson Williams. 2016. A complete dichotomy rises from the capture of vanishing signatures. *SIAM J. Comput.* 45, 5 (2016), 1671–1728.
- [9] Jin-Yi Cai, Pinyan Lu, and Mingji Xia. 2011. Computational complexity of Holant problems. *SIAM J. Comput.* 40, 4 (2011), 1101–1132.
- [10] Jin-Yi Cai, Pinyan Lu, and Mingji Xia. 2018. Dichotomy for real Holant^c problems. In *Proceedings of the SODA*. 1802–1821.
- [11] Young-Bin Choe, James G. Oxley, Alan D. Sokal, and David G. Wagner. 2004. Homogeneous multivariate polynomials with the half-plane property. *Adv. Appl. Math.* 32, 1–2 (2004), 88–187.
- [12] Jan Draisma, Dion C. Gijswijt, László Lovász, Guus Regts, and Alexander Schrijver. 2012. Characterizing partition functions of the vertex model. *J. Alg.* 350 (2012), 197–206.
- [13] Martin E. Dyer, Leslie Ann Goldberg, Catherine S. Greenhill, and Mark Jerrum. 2004. The relative complexity of approximate counting problems. *Algorithmica* 38, 3 (2004), 471–500.

- [14] Martin E. Dyer, Mark Jerrum, and Haiko Müller. 2017. On the switch Markov chain for perfect matchings. *J. ACM* 64, 2 (2017), 12:1–12:33.
- [15] Martin E. Dyer and David Richerby. 2013. An effective dichotomy for the counting constraint satisfaction problem. *SIAM J. Comput.* 42, 3 (2013), 1245–1274.
- [16] Michael Freedman, László Lovász, and Alexander Schrijver. 2007. Reflection positivity, rank connectivity, and homomorphism of graphs. *J. Amer. Math. Soc.* 20, 1 (2007), 37–51.
- [17] Andreas Galanis, Daniel Štefankovič, and Eric Vigoda. 2016. Inapproximability of the partition function for the anti-ferromagnetic Ising and hard-core models. *Comb. Probab. Comput.* 25, 4 (2016), 500–559. DOI : <https://doi.org/10.1017/S0963548315000401>
- [18] Leslie Ann Goldberg, Mark Jerrum, and Mike Paterson. 2003. The computational complexity of two-state spin systems. *Random Struct. Algor.* 23, 2 (2003), 133–154.
- [19] John H. Grace. 1902. The zeros of a polynomial. *Proc. Camb. Philos. Soc.* 11 (1902), 352–357.
- [20] Lingxiao Huang, Pinyan Lu, and Chihao Zhang. 2016. Canonical paths for MCMC: From art to science. In *Proceedings of the SODA*. 514–527.
- [21] Mark Jerrum and Alistair Sinclair. 1989. Approximating the permanent. *SIAM J. Comput.* 18, 6 (1989), 1149–1178.
- [22] Mark Jerrum and Alistair Sinclair. 1993. Polynomial-time approximation algorithms for the Ising model. *SIAM J. Comput.* 22, 5 (1993), 1087–1116.
- [23] Mark Jerrum, Alistair Sinclair, and Eric Vigoda. 2004. A polynomial-time approximation algorithm for the permanent of a matrix with nonnegative entries. *J. ACM* 51, 4 (2004), 671–697.
- [24] Mark Jerrum, Leslie G. Valiant, and Vijay V. Vazirani. 1986. Random generation of combinatorial structures from a uniform distribution. *Theor. Comput. Sci.* 43 (1986), 169–188.
- [25] Liang Li, Pinyan Lu, and Yitong Yin. 2013. Correlation decay up to uniqueness in spin systems. In *Proceedings of the SODA*. 67–84.
- [26] Chengyu Lin, Jingcheng Liu, and Pinyan Lu. 2014. A simple FPTAS for counting edge covers. In *Proceedings of the SODA*. 341–348.
- [27] Jiabao Lin and Hanpin Wang. 2018. The complexity of Boolean Holant problems with nonnegative weights. *SIAM J. Comput.* 47, 3 (2018), 798–828.
- [28] Jingcheng Liu, Alistair Sinclair, and Piyush Srivastava. 2017. The Ising partition function: Zeros and deterministic approximation. In *Proceedings of the FOCS*. 986–997.
- [29] Pinyan Lu, Menghui Wang, and Chihao Zhang. 2014. FPTAS for weighted Fibonacci gates and its applications. In *Proceedings of the ICALP*. 787–799.
- [30] Colin McQuillan. 2013. Approximating Holant problems by winding. *CoRR* abs/1301.2880 (2013).
- [31] Viresh Patel and Guus Regts. 2017. Deterministic polynomial-time approximation algorithms for partition functions and graph polynomials. *SIAM J. Comput.* 46, 6 (2017), 1893–1919.
- [32] Han Peters and Guus Regts. 2019. On a conjecture of Sokal concerning roots of the independence polynomial. *Michigan Math. J.* 68, 1 (2019), 33–55.
- [33] Guus Regts. 2018. Zero-free regions of partition functions with applications to algorithms and graph limits. *Combinatorica* 38, 4 (2018), 987–1015.
- [34] David Ruelle. 1971. Extension of the Lee-Yang circle theorem. *Phys. Rev. Lett.* 26 (1971), 303–304.
- [35] David Ruelle. 1999. Counting unbranched subgraphs. *J. Alg. Combin.* 9, 2 (1999), 157–160.
- [36] David Ruelle. 1999. Zeros of graph-counting polynomials. *Comm. Math. Phys.* 200, 1 (1999), 43–56.
- [37] Alexander Schrijver. 2013. Characterizing partition functions of the spin model by rank growth. *Indag. Math. (N.S.)* 24, 4 (2013), 1018–1023.
- [38] Alistair Sinclair, Piyush Srivastava, and Marc Thurley. 2014. Approximation algorithms for two-state anti-ferromagnetic spin systems on bounded degree graphs. *J. Stat. Phys.* 155, 4 (2014), 666–686. DOI : <https://doi.org/10.1007/s10955-014-0947-5>
- [39] Allan Sly and Nike Sun. 2014. The computational hardness of counting in two-spin models on d -regular graphs. *Ann. Probab.* 42, 6 (2014), 2383–2416.
- [40] Daniel Štefankovič, Eric Vigoda, and John Wilmes. 2018. On counting perfect matchings in general graphs. In *Proceedings of the LATIN*. 873–885.
- [41] Gábor Szegő. 1922. Bemerkungen zu einem Satz von J. H. Grace über die Wurzeln algebraischer Gleichungen. *Math. Z.* 13, 1 (1922), 28–55. DOI : <https://doi.org/10.1007/BF01485280>
- [42] Leslie G. Valiant. 2008. Holographic algorithms. *SIAM J. Comput.* 37, 5 (2008), 1565–1594.
- [43] Joseph L. Walsh. 1922. On the location of the roots of certain types of polynomials. *Trans. Amer. Math. Soc.* 24, 3 (1922), 163–180.

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