Truthful Auctions with Optimal Profit

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Abstract. We study the design of truthful auction mechanisms for maximizing the seller's profit. We focus on the case when the auction mechanism does not have any knowledge of bidders' valuations, especially of their upper bound. For the Single-Item auction, we obtain an "asymptotically" optimal scheme: for any $k \in Z^+$ and $\epsilon > 0$, we give a randomized truthful auction that guarantees an expected profit of $\Omega(\frac{OPT}{\ln OPT \ln\ln OPT \cdots (\ln^{(k)} OPT)^{1+\epsilon}})$, where OPT is the maximum social utility of the auction. Moreover, we show that no truthful auction can guarantee an expected profit of $\Omega(\frac{OPT}{\ln OPT \ln\ln OPT \cdots \ln^{(k)} OPT})$.

In addition, we extend our results and techniques to Multi-units auction, Unit-Demand auction, and Combinatorial auction.

1 Introduction

Auction has become an active area of research in Computer Science both for its commercial applications in the rapid expanding space of Internet Economy and for its algorithmic and game-theoretical appeals. A typical auction problem consists of one or more sellers who have several items to sell and a collection of bidders who want to buy what they would like to have with as little price as possible. An auction mechanism then determines who gets which items and at what price. As the participants (sellers and bidders) in an auction have their own self-incentive and private information, an auction problem can be viewed as a game among its participants.

The concept of *truthful* or *incentive compatible* mechanism captures the notion of reasonable auctions — a reasonable auction should encourage its bidders to show their true valuations. Truthfulness is a quite strong game-theoretical requirement, stating that for each bidder, bidding his/her true valuation is among the optimal strategies, no matter how other bidders behave. In another word, in a truthful auction, the decision and pricing scheme are such that there is no reason for any bidder to lie.

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1.1 Related Work and Motivations

Many auction problems have truthful mechanisms. An example is the famous Vickrey-Clarke-Groves (VCG) mechanism that maximizes the social utility [13,3,7]. However, in VCG, the maximization of the social utility might be achieved at the expense of the seller's profit — generally, the VCG scheme provides no guarantee on the seller's profit. A natural step is to design a truthful auction mechanism that maximizes profits.

Assuming that the distribution of valuations are known or can be gathered by some statistical means, VCG mechanism with a properly chosen reserved price can obtain very tight bounds on the expected profits [12,11,10]. However, there are reasons to consider profit-maximization auction without full knowledge of the valuation distributions[5].

A possible scenario is that the range of bidders' valuations is known. Given an upper bound h on the valuations, truthful auction mechanisms have been developed to achieve a profit of $\Omega\left(\frac{OPT}{\log h}\right)$, where OPT is the optimal social utility of the auction [8,9].

In absence of any valuation information, Goldberg, Hartline, Wright introduced a notion of competitive auctions in [6]. They proposed to measure the quality of the profit-maximization scheme using a worst-case competitive analysis against $F^{(2)}$, the optimal single-price auction that sells at least two items. Since then, several truthful auction schemes with constant competitive ratios have been developed [5,1,2].

Note that $F^{(2)}$ is a relatively lower bentchmark compared to OPT. In some cases, one can not bound $F^{(2)}$ with OPT. In this paper, we compare the profit directly with OPT.

1.2 Our Results

For auctions with a single item, we present a randomized truthful profit-maximization scheme and prove that it is "asymptotically" optimal. In particular, for $\forall k \in Z^+, \epsilon > 0$, we give a randomized truthful auction that guarantees an expected profit of $\Omega(\frac{OPT}{\ln OPT \ln\ln OPT \cdots (\ln^{(k)} OPT)^{1+\epsilon}})$. Moreover, we show that no truthful auction can always achieve a profit of $\Omega(\frac{OPT}{\ln OPT \ln\ln OPT \cdots \ln^{(k)} OPT})$.

Furthermore, we extend our technique for Single-Item auction to more complex auction problems such as multi-units auction, AdWords auction (Unit-Demand auction), and combinatorial auction. For multi-units and AdWords auctions, both our upper and lower bounds can be generalized. All our schemes also guarantee that the expected social utility are within a constant fraction of the optimal social utility.

For the general combinatorial auction, we build a profit-oriented auction scheme on the truthful approximation scheme of Dobzinski, Nisan, and Schapira [4]. We can achieve a profit of $\Omega(\frac{OPT}{\sqrt{m} \ln OPT \ln\ln OPT \cdots (\ln^{(k)} OPT)^{1+\epsilon}})$, where m is the number of items. When the bidders' utility functions are submodular, a profit of $\Omega(\frac{OPT}{(\log m)^2 \ln OPT \ln\ln OPT \cdots (\ln^{(k)} OPT)^{1+\epsilon}})$ can be obtained.

2 Notations

We assume that there are n bidders, and a set M of distinct items, $M = \{1, 2, \cdots, m\}$. In addition, the seller has c_j copies (c_j may be $+\infty$) of item $j \in M$. A bundle of items can be specified as a vector $(d_1, d_2, \cdots, d_m)$, where $0 \leq d_j \leq c_j, \forall j \in M$, and we denote the collection of all the bundles with $\mathcal{D}$. Each bidder i has a private valuation function v_i, which assigns a non-negative value to each bundle of items.

Each bidder submits a bid $b_i = \{b_i(S), S \in \mathcal{D}\}$. An *auction mechanism* then outputs an allocation $(S_1, S_2, \cdots, S_n)$, where $S_i \in D$, and a price $(p_1, p_2, \cdots, p_n)$. A feasible output of the mechanism must satisfy the following two conditions:

- **Limited Supply:** For each item $j \in M$, there are at most c_j copies in $(S_1, S_2, \cdots, S_n)$.
- **Individual Rationality:** For each bidder $i \in [n]$, $p_i \leq b_i(S_i)$.

A deterministic mechanism is *truthful* if for each bidder, truth-telling is a dominant strategy, which means that her utility is maximized when she bids truthfully no matter how others bid. For randomized mechanisms, there are two extensions of truthfulness, universally truthful and truthful in expectation. A randomized mechanism is *universally truthful* if it is a distribution of truthful deterministic mechanisms. *Truthfulness in expectation* means that the expected utility of a bidder is maximized when bidding truthfully.

In this paper, we focus on several special cases.

1. Single-Item auction: M consists of a single item, possibly with multiple copies. Each bidder would like to buy at most one copy.
2. Unit-Demand auction: multiple items, each item with one copy. Each bidder would like to buy at most one item and is considering a number of different options. In another words, each bidder only bids for Single-Item bundle.
3. Combinatorial auction: multiple items, each item with one copy. The bidder bids for subsets of M.

3 Single-Item Auction

In this section, we focus on the Single-Item auction. We first consider the case when there is one copy of the item, and give a profit-optimal truthful mechanism. We then extend this result to the case of multiple copies.

3.1 Single-Copy Auction

Without loss of generality, we assume the bids are $b_1 \geq b_2 \geq \cdots \geq 1$. Let $g(x) = \ln x + 1$, and $\tilde{g}^{(k)}(x) = \prod_{i=1}^{k} g^{(i)}(x)$, Recall that $g^{(i)}(x) = g(g^{(i-1)}(x)), \forall i \geq 2$.

Algorithm: `SingleCopyAuction`

INPUT: $k \in Z^{+}, \epsilon > 0$ and $\delta > 0$.

1. If there is only one bidder, we set $b_2 = 1$.
2. With probability $1 - \delta$, we use the second price auction, that is, the highest bidder wins the item at a price of the second highest price.
3. With probability δ, the seller chooses a reserved price r according to the distribution with density:

$$f_{k,\epsilon}(x) = \frac{\epsilon}{x\tilde{g}^{(k-1)}(\frac{x}{b_2})\left(g^{(k)}(\frac{x}{b_2})\right)^{1+\epsilon}}, x \in [b_2, +\infty)$$

   Then if $b_1 \geq r$, the highest bidder wins the iterm with price r. Otherwise, the item remains unsold.

It is well known that the second price auction is truthful. Because the highest bidder is the only potential recipient of the item and the reserved price r is chosen independently of her bid b_1, the auction of step 3 is a distribution of truthful mechanism. Thus, our acution scheme is universally truthful.

The algorithm above uses reserved price auction to guarantee the seller's profit while uses second price auction to enhance the social utility. The parameter δ provides a tradeoff between these two objectives.

Theorem 1 (Profit Guarantee). *Let $E(R)$ be the expected profit of the auction and $E(SU)$ be its expected social utility. Let OPT denote the maximum social utility. Then we have*

$$E(R) = \Omega\left(\frac{OPT}{\tilde{g}^{(k-1)}(OPT)\left(g^{(k)}(OPT)\right)^{1+\epsilon}}\right)$$
$$E(SU) \geq (1-\delta)OPT$$

Proof: For simplicity, we give a proof for $k = 1, \epsilon = 1$, and $\delta = \frac{1}{2}$. The proof is essentially the same for general k, ϵ, and δ.

In Single-Item auction, the optimal profit OPT is equal to the maximum bid b_1. So it is obvious that $E(SU) \geq \frac{1}{2}OPT$ because with probability of 1/2, we use the second price auction and get a social utility of OPT.

For the seller's profit, we have:

$$\begin{aligned} E(R) &= \frac{1}{2}b_2 + \frac{1}{2}\int_{b_2}^{b_1} xf(x)dx \\ &= \frac{1}{2}b_2 + \frac{1}{2}\int_{b_2}^{b_1} \frac{1}{(\ln\frac{x}{b_2}+1)^2}dx \end{aligned}$$

$$\geq \frac{1}{2}b_2 + \frac{1}{2}\frac{b_1 - b_2}{(\ln\frac{b_1}{b_2} + 1)^2}$$
$$\geq \frac{b_1}{2(\ln\frac{b_1}{b_2} + 1)^2}$$
$$\geq \frac{OPT}{2(\ln OPT + 1)^2} \qquad \square$$

We now prove that the bound in Theorem 1 is essentially tight. To show this, we first give two technical lemmas.

Lemma 1. *Let Φ be the distribution of a bid with $Pr(b_1 = 2^j) = \frac{1}{2^{j+1}}, j = 0, 1, 2, \cdots$. Then no truthful (even in expectation) mechanism can extract revenue greater than 1 on Φ.*

Proof: The distribution we used here is a modified version of distribution used in [12], and the proof is similar. $\square$

Lemma 2. *For a fixed $k \in Z^+$, $\sum_{j\geq 0}\frac{1}{\tilde{g}^{(k)}(2^j)}$ goes to infinity.*

Proof: We show that for any i, there exists constant C_i and $N_i > 0$, such that for any $x \geq N_i$, we have $g^{(i)}(x) \leq C_i \ln^{(i)} x$. This is shown by induction on i.

For $i = 1$, $g(x) = \ln x + 1 \leq 2\ln x, \forall x \geq e$.

$$\begin{aligned} g^{(i)}(x) &= g^{(i-1)}(\ln x + 1) & \\ &\leq g^{(i-1)}(2\ln x) & \text{for } x \geq e \\ &\leq C_{i-1}\ln^{i-1}(2\ln x) & \text{for } 2\ln x \geq N_{i-1} \\ &\leq C_i \ln^{(i)}(x) & \text{exists } C_i\text{, and } N_i \end{aligned}$$

For a fixed $k \in Z^+$, let $C = c_1c_2\cdots c_k$ and J be the smallest integer such that $J \in Z^+$, $2^J > max\{N_i : 1 \leq i \leq k\}$. Then we have $g^{(i)}(2^j) \leq C_i \ln^{(i)}(2^j) \leq C_i \ln^{(i-1)} j$. So we have

$$\sum_{j\geq 0}\frac{1}{\tilde{g}^{(k)}(2^j)} \geq \sum_{j=0}^{J}\frac{1}{\tilde{g}^{(k)}(2^j)} + \frac{1}{C}\sum_{j>J}\frac{1}{j\ln j\cdots\ln^{(k-1)} j} = +\infty. \qquad \square$$

Theorem 2 (Impossibility Result). *For any $k \in Z^+$, there is no truthful (even in expectation) mechanism with an expected profit of $\Omega(\frac{OPT}{\tilde{g}^{(k)}(OPT)})$.*

Proof: Assume there is a truthful auction, with an (expected) profit of $\Omega\left(\frac{OPT}{\tilde{g}^{(k)}(OPT)}\right)$. That is to say, $\exists c > 0, N > 0$, s.t. $E(R) \geq c\frac{OPT}{\tilde{g}^{(k)}(OPT)}$, when $OPT > N$. Let J be the smallest integer such that $2^J > N$. Considering the bid distribution Φ, we have

$$E(R) \geq c \sum_{j \geq J} \frac{1}{2^{j+1}} \frac{2^j}{\tilde{g}^{(k)}(2^j)}$$
$$\geq \frac{c}{2} \sum_{j \geq J} \frac{1}{\tilde{g}^{(k)}(2^j)}$$

By lemma 2, $E(R)$ goes to infinity, which contradicts with lemma 1. □

3.2 Multi-copy Auction

In a multi-copy auction, there is one item with c copies (c may be unbounded). We give a similar mechanism as Single-Copy auction. Our analysis can be extended to this case. Since there is no difference between the case $c > n$ ($c = +\infty$) and the case $c = n$, we can assume that $c \leq n$, and $b_1 \geq b_2 \geq \cdots \geq b_n$. We use the following auction scheme.

Algorithm: `MultiCopyAuction`

INPUT: $k \in Z^+, \epsilon > 0$, and $\delta > 0$.

1. If $c = n$, we set $b_{n+1} = 1$.
2. With probability $1 - \delta$, we use the VCG mechanism: sell c items to the c highest bidder at the price of the $(c+1)$-th highest bidder.
3. With probability δ we sell the items to the highest c bidders with a reserved price r chosen according to the distribution with density:

$$f_{k,\epsilon}(x) = \frac{\epsilon}{x\tilde{g}^{(k-1)}(\frac{x}{b_{c+1}}) \left(g^{(k)}(\frac{x}{b_{c+1}})\right)^{1+\epsilon}}, x \in [b_{c+1}, +\infty)$$

Similarly to the Single-Copy auction, we can obtain the following lower and upper bounds on the expected profit for multi-copy auctions.

Theorem 3. *Let OPT denote the optimal social utility and b_{max} be the highest bid (By our assumption $OPT = \sum_{j=1}^{c} b_j$ and $b_{max} = b_1$). Then we have:*

$$E(R) = \Omega\left(\frac{OPT}{\tilde{g}^{(k-1)}(b_{max})\left(g^{(k)}(b_{max})\right)^{1+\epsilon}}\right)$$
$$E(SU) \geq (1-\delta)OPT$$

In addition, no truthful auction can obtain an expected profit of $\Omega\left(\frac{OPT}{\tilde{g}^{(k)}(b_{max})}\right)$.

4 Unit-Demand Auction

We now consider the auction of multiple items, as in the keywords auction. Assume there are n bidders (advertisers), and m slots on the web page to place advertisements. The advertiser bids for each slot on the web page, and the search engine must decide which m bidders win slots, as well as the order to place the advertisements and the prices. We focus on the case when the search engine does not place two identical advertisements on the same page, which is the so-called Unit-Demand auction.

As each bidder i has a bid for each item j, we can express their valuations by a matrix $B = (b_i(j))$.

Algorithm: `AdWordAuction`

1. Choose a reserved price r according to the following distribution: with a probability of $1-\delta$, set $r = 1$; with a probability of δ, pick r according to the distribution with density
$$\frac{\epsilon}{x\tilde{g}^{(k-1)}(x)\left(g^{(k)}(x)\right)^{1+\epsilon}}, x \in [1, +\infty)$$
.
2. Compute prices $\mathbf{p}$ and allocation S by running VCG on input B with reserved prices $\mathbf{r} = (r, \cdots, r)$. The *reserved price VCG* works as follows: add m virtual bidders with bid $\mathbf{r} = (r, \cdots, r)$ into the auction, then run VCG to determine the allocation and price of each item. If an item is sold to a virtual bidder, then it is in fact unsold in the original auction.

Recall the VCG scheme for Unit-Demand auction computes a maximum weighted matching betwen bidders and items and allocates the items accordingly. The price of each item is set to be the bidding price of its recipient minus the difference of the total weights of this matching and of the maximum weighted matching without this recipient. Clearly, VCG runs in polynomial time in the number of bidders and items. Therefore, the algorithm above is a polynomial-time auction scheme.

Theorem 4. *The Unit-Demand auction is truthful and has an expected profit of* $E(R) = \Omega\left(\frac{OPT}{\tilde{g}^{(k-1)}(b_{max})\left(g^{(k)}(b_{max})\right)^{1+\epsilon}}\right)$, *where* $b_{max} = max_{i,j}\{b_i(j)\}$. *The expected social utility* $E(SU) \geq (1-\delta)OPT$.

Proof: Again for simplicity, we prove the theorem for $k = 1, \epsilon = 1$, and $\delta = \frac{2}{3}$. Let M be a maximum weighted matching between the n bidders and m items, $p_1 \geq p_2 \geq \cdots \geq p_m$ be the prices of the items sold in M, and $n_x = argmax_j\{p_j \geq x\}$. Using the similar technique in [8], we know that when the reserved price is picked at x, there are n_x items with prices higher than x sold in M, and at least half of them can be sold by the reserved price auction. So we have:

$$\begin{aligned} E(R) &= \frac{1}{3}m + \frac{2}{3}\int_1^{+\infty} \frac{n_x}{2} x f(x) dx \\ &\geq \frac{1}{3}m + \frac{1}{3}\left(\sum_{i=1}^{m} \int_{p_{i+1}}^{p_i} n_x \frac{1}{(\ln x + 1)^2} dx\right) \end{aligned}$$

$$\geq \frac{1}{3}m + \frac{1}{3}\frac{1}{(\ln p_1 + 1)^2}\sum_{i=1}^{m} i(p_i - p_{i+1})$$
$$= \frac{1}{3}m + \frac{1}{3}\frac{1}{(\ln p_1 + 1)^2}(p_1 + p_2 + \cdots + p_m - m)$$
$$\geq \frac{OPT}{3(\ln b_{max} + 1)^2}$$

With a probability of 1/3, we use the VCG with a reserved price of 1 and can obtain the optimal social utility OPT. So $E(SU) \geq \frac{1}{3}OPT$. □

5 Combinatorial Auction

We modify the algorithm in [4]. In step 3, they use a second-price auction for M, the bundle of all items, with a reserved price p_0, however, we use a randomly chosen reserved price. To be self-contained, we include the basic steps of this algorithm.

Algorithm: `CombinatorialAuction`

- **Phase I:** Partitioning the Bidders
  1. Assign each bidder to exactly one of the following three sets: SEC-PRICE with probability $1-\epsilon$, FIXED with probability $\frac{\epsilon}{2}$, and STAT with probability $\frac{\epsilon}{2}$.
- **Phase II:** Gathering Statistics
  2. Calculate the value of the optimal fractional solution in the combinatorial auction with all m items, but only with the bidders in STAT. Denote this value by OPT^*_{STAT}.
- **Phase III:** A Second-Price Auction with reserved price.
  3. Randomly pick a reserved price r according to the following density function: $f_{k,\epsilon_1}(x) = \frac{\epsilon_1}{x\tilde{g}^{(k-1)}(\frac{x}{p_0})\left(g^{(k)}(\frac{x}{p_0})\right)^{1+\epsilon_1}}, x \in [p_0, +\infty)$ where $p_0 = OPT^*_{STAT}$.
- **Phase IV:** A Fixed-Price Auction
  4. Let $R = M, p = \epsilon OPT^*_{STAT}/(8m)$.
  5. For each bidder $i \in$ FIXED, in some arbitrary order:
  (a)Let S_i be the demand of bidder i given the following prices: p for each item in R, and $+\infty$ for each item in $M - R$.
  (b) Allocate S_i to bidder i, and set his price to be $p|S_i|$.
  (c) Let $R = R \setminus S_i$.

Theorem 5. *In the general combinatorial auction,*

$$E(R) = \Omega\left(\frac{OPT}{\sqrt{m}(\tilde{g}^{(k-1)}(OPT))\left(g^{(k)}(OPT)\right)^{1+\epsilon_1}}\right)$$

When the bidders' utility functions are submodular, we have

$$E(R) = \Omega\left(\frac{OPT}{(\log m)^2(\tilde{g}^{(k-1)}(OPT))\left(g^{(k)}(OPT)\right)^{1+\epsilon_1}}\right)$$

Proof: The combinatorial auction can be formulated as a linear program. let OPT^* be the optimal fractional solution. As mentioned in [4], there are two cases:

– There is a bidder i such that $v_i(M) \geq \frac{OPT^*}{\sqrt{m}}$. This is similar to the Single-Copy auction. Let $v_{max} = max_i v_i(M)$, then $OPT \geq v_{max} \geq \frac{OPT^*}{\sqrt{m}} \geq \frac{OPT}{\sqrt{m}}$.

$$\begin{aligned} E(R) &= \Omega\left(\frac{v_{max}}{\tilde{g}^{(k-1)}(v_{max})\left(g^{(k)}(v_{max})\right)^{1+\epsilon_1}}\right) \\ &= \Omega\left(\frac{OPT}{\sqrt{m}(\tilde{g}^{(k-1)}(OPT))\left(g^{(k)}(OPT)\right)^{1+\epsilon_1}}\right) \end{aligned}$$

– For each bidder i, $v_i(M) \geq \frac{OPT^*}{\sqrt{m}}$. As shown in [4], $E(R)$ is $\Omega(\frac{OPT^*}{\sqrt{m}})$ Thus the expected profit is $\Omega\left(\frac{OPT}{\sqrt{m}(\tilde{g}^{(k-1)}(OPT))\left(g^{(k)}(OPT)\right)^{1+\epsilon_1}}\right)$.

The proof is similar for the case when the bidders' utility functions are submodular. □

6 Discussions and Future Work

In the scenario that an upper bound h of the valuations is given, we can give a mechanism which improves the profit guarantee in [8,9] by a constant factor $\log e$. The algorithm is a VCG scheme with a reserved price, which is randomly picked according to the density function $f(x) = \frac{1}{x \ln h}, x \in [1, h]$. This scheme guarantees an expected profit of $\frac{OPT}{\ln h}$, which is proved to be optimal in [12].

All the randomized VCG scheme with reserved price mentioned in our algorithms can be translated into a **Randomized-Fixed-Price Auction**. The fixed price is picked from the same distribution as that of the reserved price. Then we sell items with the fixed price to the bidders in a random order. All the profit guarantees and the proofs above still apply. Using this Randomized-Fixed-Price scheme, we can extend our results to the online auctions[1].

The Unit-Demand auction is in fact a matching problem between bidders and items. The maximum social utility are achieved by the maximum weighted matching. A natural generalization of Unit-Demand auction is the following **multi-pattern auction**: Given t_1 groups of items, the bidders have their valuations for all items. The auction mechanism then chooses one of the groups and allocates its items to the bidders.

From the view of matching, the valuations define t_1 sets of matching problems between the bidders and items. The multi-pattern auction could be useful in

Internet advertising. For example, the search engine can offer several kinds of patterns for sponsored advertising, each with several slots to place the advertisements. Each advertiser (bidder) could submit a bid for each slot in every pattern.

Assuming that there are t_1 groups and each group has t_2 items, we can extend our Unit-Demand auction scheme to obtain the following result.

Theorem 6. *For any $k \in Z^+, \epsilon > 0$, there is a truthful auction scheme with an expected profit of*

$$E(R) = \Omega\left(\frac{1}{t}\frac{OPT}{\tilde{g}^{(k-1)}(b_{max})\left(g^{(k)}(b_{max})\right)^{1+\epsilon}}\right)$$

where $b_{max} = max_{i,j}\{b_i(j)\}, t = min\{t_1, t_2\}$.

Open Problem: Can we improve the factor of $\frac{1}{t}$ in the bound?

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